Waves in Music: Applications
of Partial Differential Equations

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I. Introduction

In modern times, the idea that sound consists of waves is a generally accepted truth. In ancient times, however, theories about sound ranged from the idea of streams of atoms, proposed by Gassendi, to ray theories, in which sound travels linearly, proposed by Reynolds and Rayleigh.[1] However, evidence for the wave theory, such as diffusion of sound around corners and the detection of distinct frequencies of sound by Pythagoras (c. 550 BC) and Mersenne (1588-1648)[2], eventually led to common acceptance of the propagation of sound in wave form. Newton was the first to conduct a detailed analysis of the behavior of sound waves under various circumstances and among the first, after Mersenne, to calculate the speed of propagation of sound waves, which he called "successive pulses" of "pressure arising from vibrating parts of a tremulous body." Newton also produced a discussion of diffraction of sound waves, including Figure 1 below.[3]

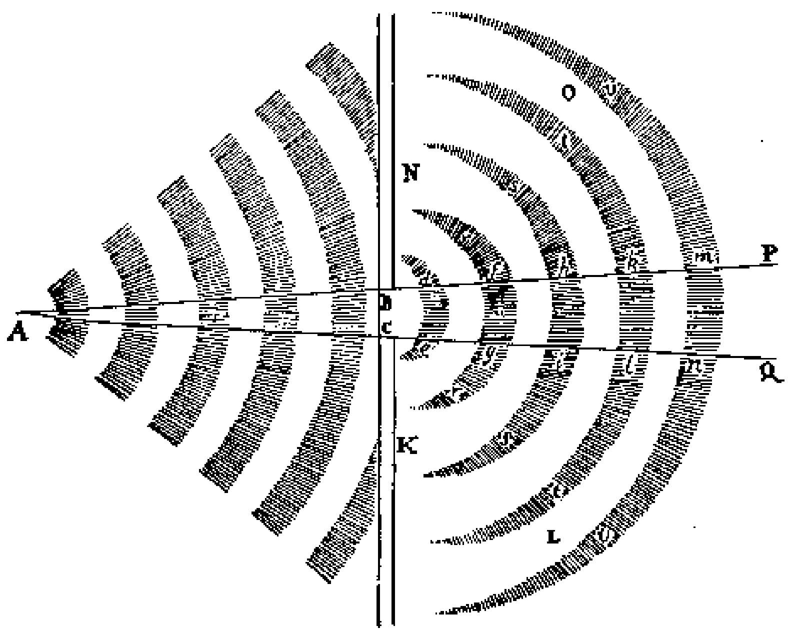

Figure 1 - Newton's Sketch of Diffraction of Waves Passing Through a Hole

[1] Pierce, 4.
[2] Ibid., 3.
[3] Sir Isaac Newton's Principia. Glasgow: MacLehose, 1871. 4th ed. p. 359. Reprinted in Pierce, 5.

Newton was aware that the amplitude of these “pulses” varied both in time and space.[4] Therefore, the obvious mathematical analysis of sound is through differential equations because sound propagates both in time and space. However, a differential equation governing wave behavior was not discovered until 1747, when d’Alembert derived the one-dimensional wave equation for a vibrating string. [5]

The significance of the mathematical behavior of waves in music is far-reaching. Without a proper understanding of the mathematics involved, it would be impossible to build a proper musical instrument or concert hall. For this reason, the theory of acoustics is closely connected to partial differential equations. The goal of this paper is therefore to make clear to the reader exactly how acoustics and PDE’s are related and to offer a discussion of various real-world complications posed by musical instruments.

II. Primary Concepts

The principle topic discussed will be the wave equation, which in its simplest form is

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad (1)$$

where ∇^2 is the n-dimensional laplacian, or sum of second derivatives with respect to the cartesian coordinates. This equation takes various forms under different circumstances, such as multiple dimensions or the presence of tension. Discovering what those forms are and applying them is the primary goal of this discussion. The derivation of this simplest wave equation for any number of dimension is rather simple from Newton’s second law, F=ma.

For the purposes of this paper, strings and surfaces such as a drumhead, a cymbal, or the surface of a hollow tube, will be considered to be negligibly thin. This is reasonable because these dimensions are very small compared to the length of the respective items in the real analogues to our

[4] Newton, 168-9.
[5] Pierce, 17. See Equation (1) in section II below.

mathematical objects. All strings, surfaces, and solids will also be considered to be absolutely smooth and homogeneous unless otherwise noted. The concepts of internal friction, interference from other objects, heat, and irregularities in forces (such as the different tension in different parts of a drumhead) are also ignored unless noted. In other words, the instruments described are assumed to be mathematically ideal lines, surfaces, and solids.

III. One Dimensional Cases

The simplest musical instrument is a vibrating string. Real examples of this include the string family (violins, violas, celli, basses, and harps) and some percussion instruments (piano, harpsichord, and dulcimers). The displacement of such a string, due to relatively high tensions, may be taken to be purely transverse, and can be represented as $y(x,t)$. We call the density per unit length $\rho(x,t)$, tension $\tau(x,t)$, and all other forces acting on the string $F(x,t)$. The total force on a string of length L is

$$\int_0^L F(x,t)dx$$

By Newton's second law, the rate of change of momentum on a segment of the string with respect to time is equal to the force on that piece:

$$\frac{\partial}{\partial t}\int_0^L \frac{\partial y(x,t)}{\partial t}\,\rho(x,t)\sqrt{1+\left(\frac{\partial y(x,t)}{\partial x}\right)^2}\;dx \;=(\tau\sin\theta)\big|_{x=x+dx}-(\tau\sin\theta)\big|_{x=x}+\int_0^L F(x,t)\,dx,$$

where θ is the angle between the tangent line to the string and the x-axis. Differentiating with respect to x gives:

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\right)=\frac{\partial}{\partial x}(\tau\sin\theta)+F(x,t)$$

For very small displacements, the radical may be dropped and $(\sin\theta)$ approximated by $\partial y/\partial x$, so this equation becomes:

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right)=\frac{\partial}{\partial x}\left(\tau\frac{\partial y}{\partial x}\right)+F(x,t).$$

With constant density and tension, this becomes:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F}{\rho}\ , \qquad c^2 = \frac{\tau}{\rho} \qquad (2),$$

which we shall call the one-dimensional wave equation, which is the model we will use to analyze a vibrating string with constant tension and mass per unit length.[6] Note that the concepts of stiffness and diameter are completely ignored here. These topics will be discussed in section VII.

It is useful to know how the wave propagates from the string to the three space dimensions, particularly if one is going to design an instrument such as a violin, in which the placement of the strings in relation to the instrument body affects the timbre, or quality, of the sound. The relevant fact is that the three-dimensional wave has roughly the form of a cylinder for large distances r from the string and varies almost entirely radially. This is a result of Huygen's treatment of wavefronts as determined by the "sphere of influence" of every point on the wavefront, shown in Figure 2.[7] Therefore, we will ignore the behavior of the wavefront very close to the source because this has little applicability to music, as the sound produced travels much further than the few wavelengths required to smooth out the wavefront somewhat before reaching the listener.

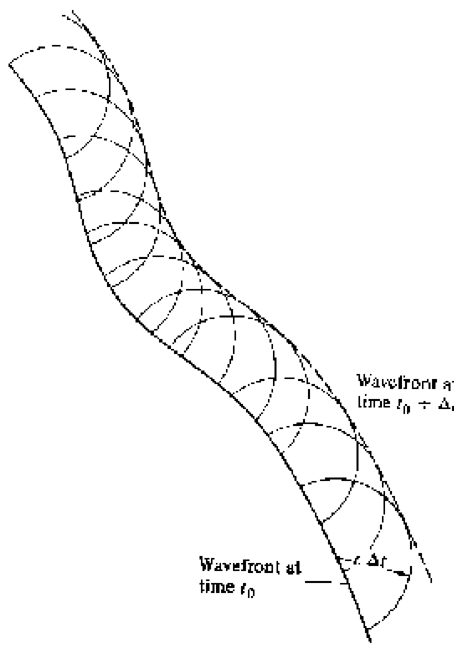

Figure 2 - Huygen's Construction of a Wavefront at a Time t+Δt from Spheres of Influence

IV. Two Dimensional Cases

The obvious next step after an analysis of one dimensional vibrations is the two-dimensional case. The two-dimensional wave equation is derived in the same manner as above, but the derivation is lengthier, so the equation is simply stated here:[8]

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{F}{\rho}, \qquad c^2 = \frac{\tau}{\rho} \qquad (3),$$

where $z(x,y,t)$ is the vertical (transverse) displacement of the point (x,y,t). This is the equation needed to analyze a vibrating flat surface, such as a drumhead with constant tension. However, because the only flat surfaces we will be considering, drumheads, are circular in shape, we need the version of (3) for polar coordinates. The derivation of this formula is rather involved[9], but follows from the transformation:

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

The equation which results is

$$\nabla^2 z = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} \qquad (4).$$

Applying the boundary conditions

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right) = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) \quad \text{on the head,}$$

$u(\mathrm{R},t) = 0$ at the rim,

$u(r,0) = \phi(r)$,

$u_t(r,0) = \psi(r)$

for an ideal drumhead yields:[10]

$$z(r,t) = \sum_{m=1}^{\infty} D_{0m} J_0(\beta_{0m} r)\sin(\beta_{0m} ct) \qquad (5)$$

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[6] Trim, 22-3.
[7] Pierce, 175.
[8] Trim, 34-5, provides a good derivation.
[9] Strauss, 151.
[10] Strauss, 253-6.

where $\beta_{nm} = \sqrt{\lambda_{nm}}$, $D_{0m} = \dfrac{\int_0^R \psi(r)\, J_0(\beta_{0m}\, r)\, r\, dr}{\frac{1}{2} R^2 c \beta_{0m} \left[J_1(\beta_{0m} R)\right]^2}$, $\psi(r) = y_t(r,0)$, and $J_n(\rho) = \sum_{j=0}^{\infty} (-1)^j \dfrac{\left(\frac{1}{2}\rho\right)^{n+2j}}{j!\,(n+j)!}$

is the Bessel function of order n. Needless to say, this is a rather unwieldy formula. Notice also that this is the solution in polar coordinates for an ideal drumhead only (mass and tension ignored), that is, a solution of (1) in polar coordinates. However, for a real drumhead, the expansion is far worse, and we will not discuss it here for brevity's sake. However, a numerical approximation would be relatively easy, even though the actual derivation of an explicit formula would not.

As in the one dimensional case, the propagation of the waves from the drumhead is very complicated, in this case involving oblate-spheroidal coordinates, shown below in Figure 3. [11]

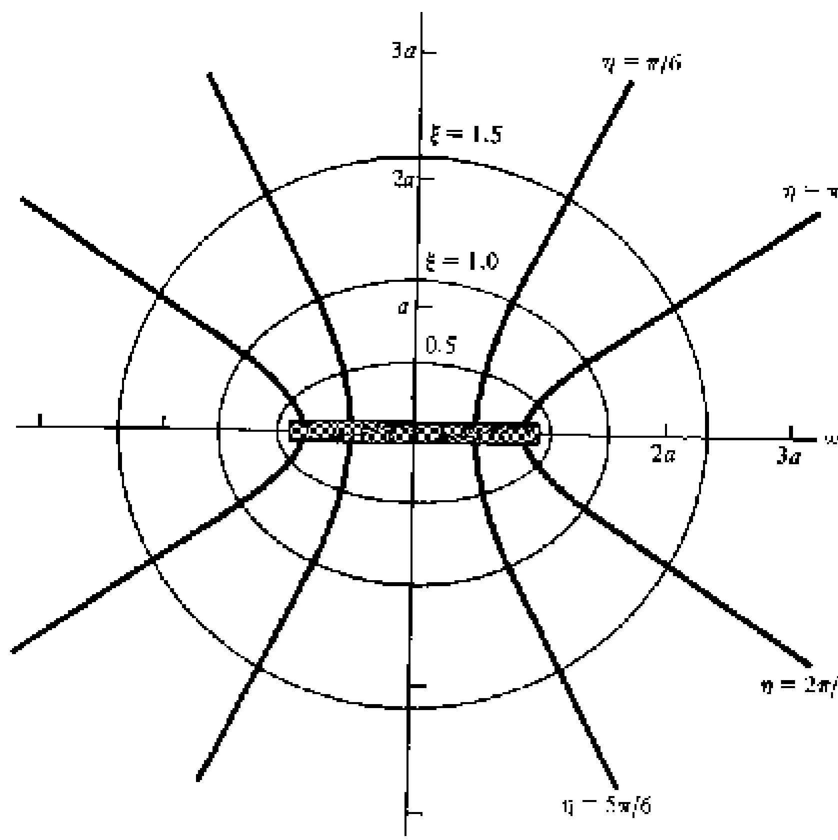

Figure 3 - Oblate Spheroidal Coordinates Around a Vibrating Disk Located in the Plane ξ=0

These coordinates, (ξ,η,φ), given by:

$$r = a \cosh \xi \sin \eta \qquad (6),$$

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$z = a \sinh \xi \cos \eta$$

where *a* the maximum radius of the drumhead, lead to the laplacian:

$$\nabla^2 u = \frac{1}{a^2 \left(\cosh^2 \xi - \sin^2 \eta\right)} \left[\frac{1}{\cosh \xi} \frac{\partial}{\partial \xi}\left(\cosh \xi \frac{\partial u}{\partial \xi}\right) + \frac{1}{\sin \eta} \frac{\partial}{\partial \eta}\left(\sin \eta \frac{\partial u}{\partial \eta}\right)\right] + \frac{1}{a^2 \cosh^2 \xi \sin^2 \eta} \frac{\partial^2 u}{\partial \varphi^2} \quad (7a),$$

which gives the wave equation:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{a^2 \left(\cosh^2 \xi - \sin^2 \eta\right)} \left[\frac{1}{\cosh \xi} \frac{\partial}{\partial \xi}\left(\cosh \xi \frac{\partial u}{\partial \xi}\right) + \frac{1}{\sin \eta} \frac{\partial}{\partial \eta}\left(\sin \eta \frac{\partial u}{\partial \eta}\right)\right] + \frac{1}{a^2 \cosh^2 \xi \sin^2 \eta} \frac{\partial^2 u}{\partial \varphi^2} \quad (7b).$$

This equation is extremely complicated to solve even under specific conditions. Therefore, we will again state the more relevant fact that the wavefront is very nearly planar over small regions at large distances and the discrepancies in pitch measurable across the width of the human eardrum are more or less insignificant.[12]

It is interesting here to mention the meaning of music in a two-dimensional universe. Huygen's principle states that, in three dimensions, waves propagate at the speed *c* and no slower. This means that the a spherical wavefront would exist as a hollow sphere in Figure 4(b), respectively, in three dimensions. By solving the corresponding wave equation (1) for the *N*th dimension, it can be shown that Huygen's principle is valid for *N*=3,5,7, . . ., but not for *N* even. The domain of dependence and "sphere" of influence for an even-numbered dimension therefore include both the surface and the interior of the "sphere." What this means is that a two dimensional symphony would be heard as an overlapping mix of what the flat musicians were playing and everything they had played before. This plays havoc with wavelengths and frequencies, as well, and also applies to light. A two-dimensional concert-goer would be deaf and blind before he or she figured out what he or she was looking at and found the concert hall.[13]

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[11] Diagram from Pierce, 191.

[12] Pierce, 191-195, discusses both these coordinates and the solution of Laplace's equation on the drum ((7a) = 0) in greater detail.

[13] Strauss, 39-40, 222-227, provide more detailed discussions of this topic.

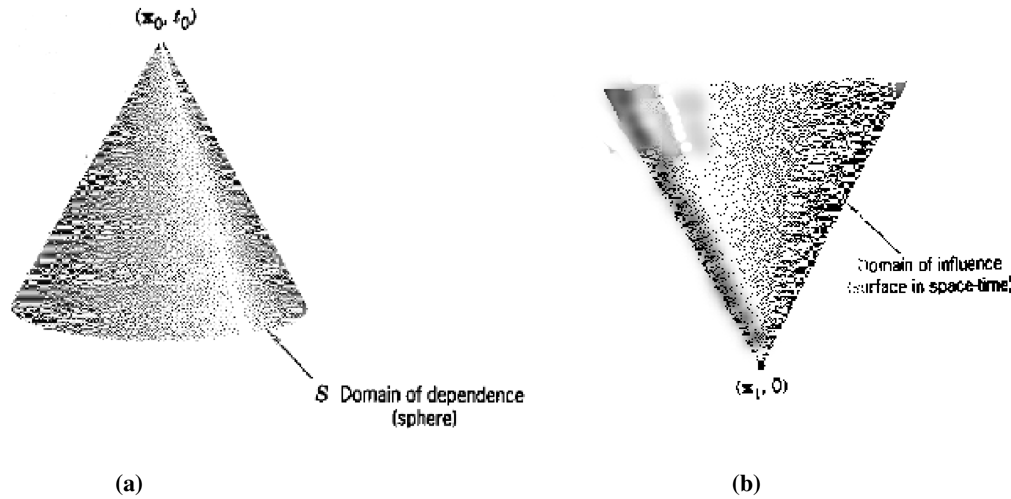

Figure 4 - (a) The Domain of Dependence in Space-Time (Spherical), (b) The Domain of Influence (Surface in Space-Time)[14]

V. Three Dimensional Solids

The first three-dimensional case we will consider is the simplest, a uniform rectangular solid. Although the spherical wave equation can be derived in a manner similar to equation (5) above, it has no relevance to musical applications, so we will not discuss it here. The derivation of the three-dimensional wave equation in cartesian coordinates follows almost directly from the derivation of the two-dimensional wave equation. Therefore, it is simply stated here:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{F}{\rho} \qquad c^2 = \frac{\tau}{\rho} \qquad (8).$$

Note that in three dimensions, F represents not tension of a string or drumhead but instead the attractive forces that hold the solid together. These forces include gravity, intermolecular attractions, and bonding forces in the real world.

VI. Three Dimensional Curved Surfaces

The case of a thin three dimensional curved surface, such as a cymbal, requires a more complicated analysis than the other cases we have looked at. In the first two, "straight," cases we have looked at, the normal vector was simply another space vector. Because the line and drumhead were not curved when at rest, all wave motion occurred in the additional space dimension. However, this is not the case for a curved surface in three dimensions. Here the normal vector depends on the shape of the surface, and, therefore, so does the direction of displacement in wave motion. The result is that the differential equation depends on the gradient on the "at rest" surface, and therefore depends on multiple derivatives. Because of these difficulties, no method for solving the equation on such a surface implicitly exists.

It is possible to derive an equation for the radiation of waves from such a body oscillating at a constant frequency, known as the Kirchhoff-Helmholtz integral theorem.[15] The resulting formula gives the amplitude of the acoustic pressure at a point (x', y', z') as a double integral in terms of the amplitude of vibration on the surface of the body at some time, the frequency of oscillation, and the distance of the point (x', y', z') from the point (x_s, y_s, z_s) on the surface. Without the solution on the surface itself, though, this is a relatively useless formula. The solution does have some application to the real drumhead of finite thickness as discussed in section IV, but, as we will discuss later, such a solution is complicated by further real-world conditions.

VII. Applications of PDE's in Music

The purpose of this discussion has been to analyze the application of the wave equation to music, which is what we will now finally address. As stated before, equation (1) or some variation thereof governs all wave behavior, which means all forms of sound propagation. Therefore, the proper design of musical instruments, concert halls, and, for that matter, any room or device intended to produce or absorb sound all depend on an understanding of the principles behind and resulting from the wave equation. The variations of the wave equation presented here are particularly

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[14] Figures from Strauss, 223.
[15] Pierce, 180-90, deals with this in greater detail.

useful in these endeavors, but primarily so in the field of musical instrument design. The variations which cover strings, level surfaces, curved surfaces, and solids encompass all musical instruments currently in existence.

Equation (2), which governs the wave behavior of a string, is applicable in the design of instruments in the string family, including violins, violas, celli, basses, guitars, harps, pianos, and dulcimers. One aspect of real string behavior neglected in the above discussion is stiffness, which in a real string introduces a shear force and bending moment. The equation governing such strings, which we will not derive here, is:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} - \alpha\left(\frac{\partial^4 u}{\partial x^4}\right)\right] ,$$

where α is a constant depending on cross-section and elasticity.[16] Another issue is the driving frequency of a string. While piano, harp, and dulcimer strings are usually struck or plucked only once, the violin family instruments, although they may be plucked, are more often bowed. As a result, the bow, which also resonates, provides a driving frequency which may differ from the resonant frequency of the string. This is a rather detailed topic and must be omitted here. The basic result, though, is that the strings of the instrument oscillate at a frequency which eventually approaches the natural frequency of the driver, in this case, the bow.[17]

The case of a flat surface is applicable in the design of the drumhead, which is, of course, also our model. Stiffness also plays a role in the two-dimensional case, but such a discussion is relatively pointless without taking variable tension into account, as well. All real drumheads and most other mechanical equivalents are tuned not with constant tension, but by use of a rim and series of lugs, which put more pressure on the rim of the head near the lugs than elsewhere. The result is a tension which varies periodically with respect to the θ variable, with a period of $2\pi/n$, n being the number of lugs on a particular drum, usually 6 to 12. Because the conditions on the real drumhead are far more complicated than those on our ideally thin head with constant tension, the corresponding solutions to the appropriate wave equation are far more complicated than the already complex equation (5).

[16] Main, 213-5, provides the derivation in different notation.
[17] Ibid., 56-75, provides a full discussion of driven oscillations.

However, for most drums the tension is close enough to constant and the thickness is so small compared to the other space dimensions that these perturbations are insignificant.

The only direct musical analogues to the solid in cartesian coordinates described in equation (8) are the keys of the keyboard percussion instruments, namely the xylophone, concert bells or glockenspiel, marimba, and vibraphone. These objects behave in a manner very similar to a string of equal length and oscillate directly up and down with a fairly constant frequency. While all of the other cases discussed are, in fact, governed by some variation of equation (8), these cases are handled separately and no detailed analysis of section V is necessary.

The final case, section VI, is the case most applicable to the wind instruments and the bodies of the stringed instruments. All members of the brass and woodwind families are essentially very thin sheets of metal or wood bent or cut in various shapes, most often basically cylindrical or shaped like a flared cylinder. In fact, the violin family instruments, which consist of thin sheets of wood, and the shells of drums, which are thin sheets of metal, are also governed by the topics of section VI. An understanding of at least the physical principles resulting from such surfaces, if not the mathematics itself, is essential for anyone contemplating designing a musical instrument. Modern wind instruments are made primarily as though they were tubular or conical in shape, and the keys or valves are placed by combinations of physics and trial and error. It is extremely difficult to produce an instrument that reproduces a musical scale exactly due to physical effects of the instrument material and the slight irregularity of the musical scale. The primary difference, in fact, between inexpensive horns and so-called “professional” models is the set of tolerances to which the physics is applied. The issue of the vibrations within a box such as a violin body or within a concert hall rather than on the respective surfaces are more issues for the field of acoustics,[18] but the mathematics is firmly rooted in partial differential equations.

Further complications of every case described here include friction, whether from other objects or from the surrounding air, and heat. The equations here all implicitly assume constant

[18] Pierce and Main, 299-318, provide discussions of these and related topics.

temperature throughout the object of interest. In reality, though, heat is produced by the player's hands, by friction in the case of the string and percussion instruments, and by the player's breath in the wind instruments. The introduction of variable heat means that the term c is no longer constant. This leads to combinations of the wave and diffusion equations, the diffusion equation being:

$$\frac{\partial u}{\partial t} = \mathrm{k}\, \nabla 2 u \ .$$

In other words, the wave equation on a real musical instrument must be a hybrid equation involving at least two time derivatives and at least the three second-order space derivatives. At present, only approximation methods[19] can be used for such problems, because only extremely special cases yield equations which can be solved explicitly using classical theories of partial differential equations.

VIII. Conclusions

Based on our analysis, we conclude that the wave equation and its variants are present in every aspect of sound wave production and propagation. Therefore, an understanding of at least the basic variations on the wave equation and their respective derivations is essential to anybody studying or working in fields involve waves. Further research in these areas would lead to a more detailed study of the cases presented here, especially those in section VII. The only currently known methods of dealing with such complicated equations as the Kirchhoff-Helmholtz integral theorem or the doubly time-dependent equation discussed at the end of section VII in non-trivial cases are numerical analysis methods, particularly the finite element method.[20] Further research needs to be done in finding explicit solutions for those equations which now have only approximations. The solutions of the Kirchhoff-Helmholtz integral formula include the radiation of waves from any body

[19] See section VIII below.

[20] Strauss, 212-4, provides an introduction to this method. A number of references dealing solely with this method exist.

and therefore are highly applicable to musical instrument design and concert hall design. The issue addressed above of temperature variation is also a consideration in the real world. With solutions to the appropriate equations in, on, and around a musical instrument, it may one day be possible to design instruments which require no compensation to remain in tune on different notes or concert halls designed to accentuate the range and timbre of a specific instrument in a specific location. However, such equations are extremely complex and must be left to a far more detailed discussion.

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