7 Difference Equations

Many of the sequences we will study in this course can be expressed as difference equations:

Def: Let $z(\cdot)$ and $w(\cdot)$ be sequences of real numbers. Then

$$z(t) + \alpha_1 z(t-1) + \cdots + \alpha_p z(t-p) = w(t)$$

is called a difference equation of order p, coefficients $\alpha_1, \ldots, \alpha_p$, and forcing term $w(\cdot)$. If $w(t) = 0$ for all t, we say the difference equation is homogeneous.

7.1 Recursive Calculation of Values of Difference Equations

To calculate the values of z for all values of t it is sufficient to know all of the values of w and any p consecutive values of z. These p values are called starting values or initial conditions. If we know the starting values $z(1), \ldots, z(p)$ and the values $w(p+1), \ldots, w(n)$ for a difference equation, then we can find $z(p+1), \ldots, z(n)$ recursively by

$$z(p+j) = w(p+j) - \sum_{k=1}^{p} \alpha_k z(p+j-k), \qquad j = 1, \ldots, n-p.$$

Note that $w(p+1)$ and $z(1), \ldots, z(p)$ are used to find $z(p+1)$, which is in turn used in finding $z(p+2)$, and so on.

7.2 Solving Difference Equations

There are two things we would like to do when we have a difference equation:

1. **Solve it:** We would like an explicit formula for $z(t)$ that is only a function of t, the coefficients of the difference equation, and the starting values.
2. **Study the asymptotic behavior of $z(t)$:** We would like to know what happens to the value of $z(t)$ as $t \to \infty$.

Ex: The homogeneous 1st order difference equation

$$z(t) + \alpha_1 z(t-1) = 0, \quad t \geq 1,$$

can be solved recursively

$$\begin{aligned} z(t) &= -\alpha_1 z(t-1) = -\alpha_1[-\alpha_1 z(t-2)] \\ &= \cdots = [-\alpha_1]^t z(0), \end{aligned}$$

that is,

$$z(t) = [-\alpha_1]^t z(0),$$

which is a solution to the difference equation. Further, from this expression for $z(t)$, we can see that

1. If $|\alpha_1| < 1$, then $z(t) \to 0$.

2. If $\alpha_1 = 1$, then $z(t)$ alternates between $\pm z(0)$ for all t, while if $\alpha_1 = -1$, then $z(t) = z(0)$ for all t.

3. If $\alpha_1 < -1$, then $z(t) \to \infty$.

4. If $\alpha_1 > 0$, then $z(t)$ alternates between positive and negative values that are getting larger and larger in absolute value.

7.3 Solving a Homogeneous Difference Equation

Consider the polynomial $h(z) = \sum_{j=0}^{p} \alpha_j z^{p-j}$ (called the indicial polynomial of the DE). Since the coefficients are real, we know that a zero of h is either real or if complex then its complex conjugate must be a zero as well.

Solution: The solution of a homogeneous DE is the sum of terms of two types (we discuss below how to find the β's, γ's, and δ's in the terms):

1. If z is a real zero of h that occurs m times, then there is a term of the form
$$\left(\beta_1 + \beta_2 t + \cdots + \beta_m t^{m-1}\right) z^t,$$
which is $\beta_1 z^t$ if $m = 1$.

2. If $z = a + bi$ is a complex zero (and thus $\bar{z} = a - bi$ is also a zero) and z and $\bar{z}$ occur m times each, then there is a term of the form
$$\left[\gamma_1 \cos(t\theta+\delta_1)+\gamma_2 t \cos(t\theta+\delta_2)+\cdots+\gamma_m t^{m-1} \cos(t\theta+\delta_m)\right] r^t,$$
where $r = |z|$ and $\theta = \arctan(b/a)$. This term is $\gamma_1 \cos(t\theta + \delta_1) r^t$ if $m = 1$.

The β's, γ's and δ's are determined by solving for them in the p equations given by setting the initial conditions equal to the expression for the solution.

7.4 Examples of Solving a Difference Equation

Ex1: If we have

$$z(t) - z(t-1) + 0.25z(t-2) = 0, \quad t \geq 2,$$

with $z(0) = 1$ and $z(2) = 3$, we have

$$h(z) = z^2 - z + 0.25 = (z - .5)^2,$$

which has the zero 0.5 occurring twice. So we have a single term in the solution of the form

$$(\beta_1 + \beta_2 t)0.5^t,$$

and substituting the starting values into this expression gives

$$z(0) = 1 = \beta_1, \quad z(1) = 3 = 0.5(\beta_1 + \beta_2),$$

which gives $\beta_1 = 1$ and $\beta_2 = 5$, so the solution is

$$z(t) = (1 + 5t)0.5^t, \quad t \geq 0.$$

Notice that because the zero of h was less than one in absolute value, $z(t) \to 0$ as $t \to \infty$.

Ex2: If we have for a fixed ω

$$z(t) - 2\cos(\omega)z(t-1) + z(t-2), \quad t \geq 2,$$

with $z(0) = 1$ and $z(1) = \cos(\omega)$, then

$$h(z) = z^2 - 2\cos(\omega)z + 1,$$

which has zeros $\cos(\omega) \pm i\sin(\omega) = e^{\pm i\omega}$, so $r = 1$ and $\theta = \omega$, which gives the solution as

$$z(t) = \gamma_1 \cos(t\omega + \delta_1)1^t,$$

and substituting the starting values gives

$$z(0) = 1 = \gamma_1 \cos(\delta_1), \quad z(1) = \cos(\omega) = \gamma_1 \cos(\omega + \delta_1),$$

which is satisfied by

$$\gamma_1 = 1, \quad \delta_1 = 0,$$

which gives

$$z(t) = \cos(\omega).$$

Thus the asymptotic behavior here is to continue the cosine.

7.5 The Behavior of the Solutions

The two examples above illustrate some of the ways that difference equations can behave depending on the zeros of h; whether they are real or complex and whether they are greater than one, equal to one, or less than one in modulus.

In general we can say something like if the zeros are all real and distinct, then as $t \to \infty$, the values of $z(t)$ either converge to zero or some other constant, or oscillate back and forth between constants, or oscillate between $-\infty$ and ∞, or diverge to ∞.

If we have complex zeros, then the difference equation can appear sinusoidal or a combination of a polynomial plus or times a sinusoid.