Chapter 4

Duality and Sensitivity Analysis

4.1 INTRODUCTION

The optimal solution of a linear-programming problem represents a snapshot of the conditions that prevail at the time the model is formulated. In the real world, decision environments rarely remain static, and it is essential to equip LP with the capability to determine changes in the optimal solution that result from making changes in the parameters of the model. This is what *sensitivity analysis* does. It provides efficient computational techniques that allow us to study the dynamic behavior of the optimal solution.

We have already dealt with the topic of sensitivity analysis at an elementary level in Section 2.4. In this chapter, we present an algebraic treatment of the topic based on the use of *duality theory*.

Sensitivity analysis deals with making *discrete* changes in the parameters of the model. A generalization of this situation is the case in which the parameters change according to predetermined *continuous* functions. The technique, called *parametric programming,* will be presented in Section 7.7.

4.2. DEFINITION OF THE DUAL PROBLEM

The LP model we develop for a situation is referred to as the **primal** problem. The **dual** problem is a closely related mathematical definition that can be derived directly from the primal problem. This section provides the mathematical details of the dual model.

In most LP treatments, the dual is defined for various forms of the primal depending on the sense of optimization (maximization or minimization), the types of the constraints ($\leq$, $\geq$, and $=$), and the sign of the variables (nonnegative or unrestricted). This type of treatment may be confusing (see Problem 4.2a–7). In this book, we present a *single* definition that automatically subsumes *all* the forms of the primal. The definition assumes that the primal problem is expressed in the *standard form* (Section 3.2), which is defined as

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, m$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, n$$

The variables $x_j, j = 1, 2, \ldots, n$, include the surplus and slacks, if any. The standard form has three properties.

1. All the constraints are equations (with nonnegative right-hand side).
2. All the variables are nonnegative.
3. The sense of optimization may be maximization of minimization.

Remember that the standard form is always used to produce the starting tableau of the simplex method and that the solution of the dual problem can be obtained directly from the optimal primal simplex tableau, as will be shown in Section 4.3. Thus, by defining the dual from the standard primal, we automatically obtain a dual solution that is consistent with the simplex method computations.

The variables and constraints of the dual problem can be constructed symmetrically from the primal problem as follows:

1. A dual variable is defined for each of the m primal constraint equations.
2. A dual constraint is defined for each primal of the n primal variables.
3. The left-hand–side coefficients of the dual constraint equal the constraint (column) coefficients of the associated primal variable. Its right-hand side equals the objective coefficient of the same primal variable.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

Table 4–1 summarizes this information pictorially with $y_1, y_2, \ldots$, and y_m representing the dual variables.

The rules for determining the sense of optimization, the type of the constraint, and the sign of the variables in the dual problem are summarized in Table 4–2. The following examples demonstrate the implementation of these rules.

TABLE 4–1

	Primal variables						
	x_1	x_2	...	x_j	...	x_n	
Dual variables	c_1	c_2	...	c_j	...	c_n	
y_1	a_{11}	a_{12}	...	a_{1j}	...	a_{1n}	b_1
y_2	a_{21}	a_{22}	...	a_{2j}	...	a_{2n}	b_2
.	.	.		.		.	.
.	.	.	...	.	...	.	.
.	.	.		.		.	.
y_m	a_{m1}	a_{m2}	...	a_{mj}	...	a_{mn}	b_m
				jth Dual constraint			Dual objective

TABLE 4–2

Standard primal problem objective	Dual problem: Objective	Constraints type	Variables sign
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

Example 4.2–1.

Primal	Standard primal	Dual variables
Maximize $z = 5x_1 + 12x_2 + 4x_3$ subject to	Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$ subject to	
$x_1 + 2x_2 + x_3 \leq 10$	$x_1 + 2x_2 + x_3 + x_4 = 10$	y_1
$2x_1 - x_2 + 3x_3 = 8$	$2x_1 - x_2 + 3x_3 + 0x_4 = 8$	y_2
$x_1, x_2, x_3 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Minimize } w = 10y_1 + 8y_2$$

subject to

$$y_1 + 2y_2 \geq 5$$

$$2y_1 - y_2 \geq 12$$

$$y_1 + 3y_2 \geq 4$$

$$\left.\begin{array}{r} y_1 + 0y_2 \geq 0 \\ y_1, y_2 \text{ unrestricted} \end{array}\right\} \Rightarrow (y_1 \geq 0, y_2 \text{ unrestricted})$$

Example 4.2–2.

Primal	Standard primal	Dual variables
Minimize $z = 15x_1 + 12x_2$ subject to $x_1 + 2x_2 \geq 3$ $2x_1 - 4x_2 \leq 5$ $x_1, x_2 \geq 0$	Minimize $z = 15x_1 + 12x_2 + 0x_3 + 0x_4$ subject to $x_1 + 2x_2 - x_3 + \quad = 3$ $2x_1 - 4x_2 \quad + x_4 = 5$ $x_1, x_2, x_3, x_4 \geq 0$	 y_1 y_2

Dual Problem

$$\text{Maximize } w = 3y_1 + 5y_2$$

subject to

$$y_1 + 2y_2 \leq 15$$

$$2y_1 - 4y_2 \leq 12$$

$$-y_1 \leq 0 \quad (\text{or } y_1 \geq 0)$$

$$y_2 \leq 0$$

$$y_1, y_2 \text{ unrestricted (redundant)}$$

Example 4.2–3.

Primal	Standard primal	Dual variables
Maximize $z = 5x_1 + 6x_2$ subject to $x_1 + 2x_2 = 5$ $-x_1 + 5x_2 \geq 3$ $4x_1 + 7x_2 \leq 8$ x_1 unrestricted $x_2 \geq 0$	Substitute $x_1 = x_1^+ - x_1^-$, to get Maximize $z = 5x_1^+ - 5x_1^- + 6x_2$ subject to $x_1^+ - x_1^- + 2x_2 \quad = 5$ $-x_1^+ + x_1^- + 5x_2 - x_3 \quad = 3$ $4x_1^+ - 4x_1^- + 7x_2 \quad + x_4 = 8$ $x_1^+, x_1^-, x_2 \geq 0$	 y_1 y_2 y_3

Dual Problem

$$\text{Minimize } z = 5y_1 + 3y_2 + 8y_3$$

subject to

$$\left.\begin{aligned} y_1 - y_2 + 4y_3 &\geq 5 \\ -y_1 + y_2 - 4y_3 &\geq -5 \end{aligned}\right\} \Rightarrow (y_1 - y_2 + 4y_3 = 5)$$

$$2y_1 + 5y_2 + 7y_3 \geq 6$$

$$-y_2 \geq 0 \Rightarrow (y_2 \leq 0)$$

$$y_3 \geq 0$$

y_1 unrestricted

y_2, y_3, unrestricted (redundant)

The first and second constraints are replaced by an equation. The rule in this case is that an unrestricted primal variable always corresponds to an equality dual constraint, and conversely, a primal equation produces an unrestricted dual variable.

Problem set 4.2a

1. In Example 4.2–1, derive the associated dual problem if the sense of optimization in the primal problem is changed to minimization.
2. Consider Example 4.2–1. The application of the simplex method to the primal requires the use of an artificial variable in the second constraint of the standard primal to secure a starting basic solution. Show that the presence of an artificial primal variable does not affect the definition of the dual because it leads to a redundant dual constraint.
3. In Example 4.2–2, derive the associated dual problem given that the primal problem is augmented with the third constraint $3x_1 + x_2 = 4$.
4. In Example 4.2–3, show that even if the sense of optimization in the primal is changed to minimization, an unrestricted primal variable always corresponds to an equality dual constraint.
5. Write the dual for each of the following primal problems:

   (a) Maximize $z = -5x_1 + 2x_2$
   subject to

   $$-x_1 + x_2 \leq -2$$

   $$2x_1 + 3x_2 \leq 5$$

   $$x_1, x_2 \geq 0$$

   (b) Minimize $z = 6x_1 + 3x_2$
   subject to

Maximization Problem		Minimization Problem
Constraints		Variables
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	Unrestricted
Variables		Constraints
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

4.3 RELATIONSHIP BETWEEN THE OPTIMAL PRIMAL AND DUAL SOLUTIONS

The primal and dual problems are so closely related that the optimal solution of one problem can be secured directly (without further computations) from the optimal simplex tableau of the other problem. This result is based on the following property:

Property I. *At any simplex iteration of the primal or the dual,*

$$\begin{pmatrix} \text{Objective coefficient} \\ \text{of variable } j \text{ in} \\ \text{one problem} \end{pmatrix} = \begin{pmatrix} \text{Left-hand side } \textit{minus} \\ \text{right-hand side} \\ \text{of constraint } j \text{ in} \\ \text{the other problem} \end{pmatrix}$$

The property is symmetrical with respect to both the primal and the dual problems.

Property I can be used to determine the optimal solution of one problem (directly) from the optimal simplex tableau of the other. This result could be advantageous computationally if the computations associated with the solved problem is considerably less than those associated with the other problem. For example, if a model has 100 variables and 500 constraints, it is advantageous computationally to solve the dual because it has only 100 constraints.

Example 4.3–1.

Consider the primal and dual problems of Example 4.2–1, which are repeated here for convenience.

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3$ subject to $x_1 + 2x_2 + x_3 \le 10$ $2x_1 - x_2 + 3x_3 = 8$ $x_1, x_2, x_3 \ge 0$	Minimize $w = 10y_1 + 8y_2$ subject to $y_1 + 2y_2 \ge 5$ $2y_1 - y_2 \ge 12$ $y_1 + 3y_2 \ge 4$ $y_1 \ge 0, y_2$ unrestricted

The following tableaux provide the simplex iterations for the primal problem.

Iteration	Basic	x_1	x_2	x_3	x_4	R	Solution
	z	$-5-2M$	$-12+M$	$-4-3M$	0	0	$-8M$
0	x_4	1	2	1	1	0	10
	R	2	-1	3	0	1	8
	z	$-\frac{7}{3}$	$-\frac{40}{3}$	0	0	$\frac{4}{3}+M$	$\frac{32}{3}$
1	x_4	$\frac{1}{3}$	$\frac{7}{3}$	0	1	$-\frac{1}{3}$	$\frac{22}{3}$
	x_3	$\frac{2}{3}$	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$\frac{8}{3}$
		$-\frac{3}{7}$	0	0	$\frac{40}{7}$	$-\frac{4}{7}+M$	$\frac{368}{7}$
2	x_2	$\frac{1}{7}$	1	0	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{22}{7}$
	x_3	$\frac{5}{7}$	0	1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{26}{7}$
	z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5}+M$	$\frac{274}{5}$
3	x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
	x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Applying Property I to the starting solution variables x_4 and R in optimal iteration 3, we obtain the following information:

Starting primal variables	x_4	R
z-Equation coefficient (iteration 3)	$\frac{29}{5}$	$-\frac{2}{5}+M$
Associated dual constraint	$y_1 \ge 0$	$y_2 \ge -M$
Equation resulting from Property I	$y_1 - 0 = \frac{29}{5}$	$y_2 - (-M) = -\frac{2}{5} + M$

The solution of the given equations yields $y_1 = \frac{29}{5}$ and $y_2 = -\frac{2}{5}$. If you solve the dual problem independently, you will obtain the same solution. Also, because of the symmetry of Property I with respect to the primal and the dual problem, a similar application to the starting variables in the optimal dual tableau will automatically yield the optimal primal solution $x_1 = \frac{26}{5}, x_2 = \frac{12}{5}$, and $x_3 = 0$ (use TORA to solve the dual and verify that the given assertion is true).

The application of Property I to the starting variables always results in easy-to-solve equations because each equation involves exacty one variable. Nothing, however, should prevent us from using any two of the other (primal) variables (that is, x_1, x_2, and x_3) to generate the desired equations. For example, at the optimal tableau, Property I equations associated with x_1 and x_3 are, respectively,

$$y_1 + 2y_2 - 5 = 0$$

$$y_1 + 3y_2 - 4 = \frac{3}{5}$$

The solution of these two equations still yields the same optimal dual values $y_1 = \frac{29}{5}$ and $y_2 = -\frac{2}{5}$. However, the equations are not as simple as those associated with x_4 and R (convince yourself that any two of the variables x_1, x_2, x_3, x_4, and R will produce the dual variables).

We present next a relationship between the primal and the dual, which, together with Property I, can be used to provide interesting economic interpretations of the linear programming problem.

Property II. *For any pair of feasible primal and dual solutions,*

$$\begin{pmatrix}\text{Objective value in the}\\ \textit{maximization}\text{ problem}\end{pmatrix} \leq \begin{pmatrix}\text{Objective value in the}\\ \textit{minimization}\text{ problem}\end{pmatrix}$$

At the optimum, the relationship holds as a strict equation.

Example 4.3–2.

In Example 4.2–1, the primal and dual problems can be shown (by inspection of the constraints) to have the feasible solutions ($x_1 = 0, x_2 = 0, x_3 = \frac{8}{3}$) and ($y_1 = 6, y_2 = 0$). The associated values of the objective functions are then given as $z = 10\frac{2}{3}$ and $w = 60$. Conversely, the optimum solution for the two problems ($x_1 = \frac{26}{5}, x_2 = \frac{12}{5}, x_3 = 0$) and ($y_1 = \frac{29}{5}, y_2 = -\frac{2}{5}$) yield $z = w = 54.8$. Both calculations demonstrate the property.

Property II reveals that for all feasible solutions of the primal and the dual, the objective value in the minimization problem always provides an upper bound on the objective value of the maximization problem. Given that the successive iterations of the maximization problem will result in increasing the value of the objective function, and those of the minimization problem will result in decreasing the value of the objective function. Eventually, in the course of the successive iterations, an equilibrium point will be reached where the maximization and the minimization objective values must be equal.