4. The Dual Simplex Method

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Problem Solving and Constraint Programming (RPAR)

Basic Idea (1)

- Algorithm as explained so far known as primal simplex:
  starting with feasible basis,
  look for optimal basis while keeping feasibility

- Alternative algorithm known as dual simplex:
  starting with optimal basis,
  look for feasible basis while keeping optimality

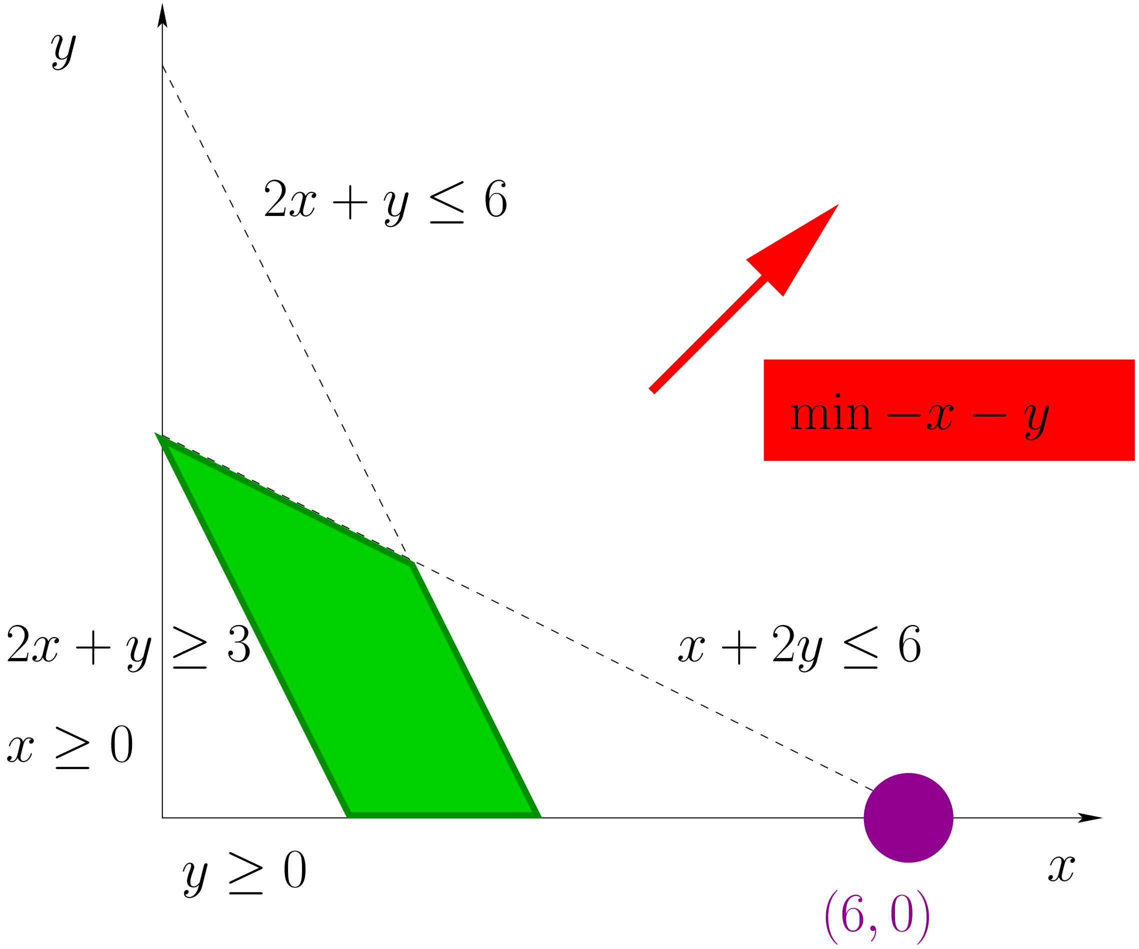
Basic Idea (2)

$$\begin{cases} \min \; -x - y \\ 2x + y \geq 3 \\ 2x + y \leq 6 \\ x + 2y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases} \implies \begin{cases} \min \; -x - y \\ 2x + y - s_1 = 3 \\ 2x + y + s_2 = 6 \\ x + 2y + s_3 = 6 \\ x, y, s_1, s_2, s_3 \geq 0 \end{cases}$$

$$\begin{cases} \min -6 + y + s_3 \\ x = 6 - 2y - s_3 \\ s_1 = 9 - 3y - 2s_3 \\ s_2 = -6 + 3y + 2s_3 \end{cases}$$

Basis (x, s_1, s_2) is optimal but not feasible!

Basic Idea (3)



Basic Idea (4)

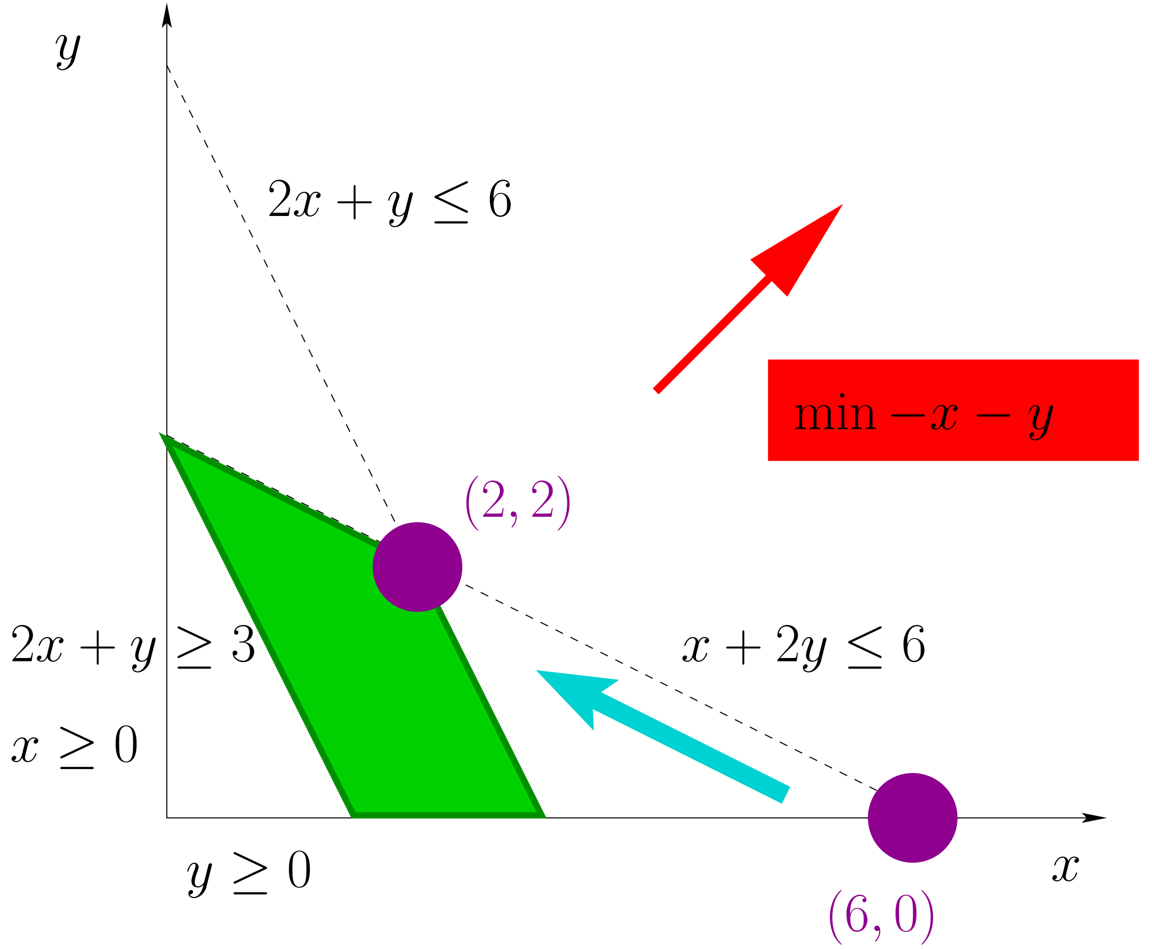
- Let us make the violating variables non-negative ...
  - Increase s_2 by making it non-basic
- ... while preserving optimality
  - If y replaces s_2 in the basis,
    then $y = \frac{1}{3}(s_2 + 6 - 2s_3), -x - y = -4 + \frac{1}{3}(s_2 + s_3)$
  - If s_3 replaces s_2 in the basis,
    then $s_3 = \frac{1}{2}(s_2 + 6 - 3y), -x - y = -3 + \frac{1}{2}(s_2 - y)$
  - To preserve optimality, y must replace s_2

Basic Idea (5)

$$\left\{\begin{array}{l} \min -6 + y + s_3 \\ x = 6 - 2y - s_3 \\ s_1 = 9 - 3y - 2s_3 \\ s_2 = -6 + 3y + 2s_3 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} \min\ -4 + \frac{1}{3}s_2 + \frac{1}{3}s_3 \\ x = 2 - \frac{2}{3}s_2 + \frac{1}{3}s_3 \\ y = 2 + \frac{1}{3}s_2 - \frac{2}{3}s_3 \\ s_1 = 3 - s_2 \end{array}\right.$$

- Current basis is feasible and optimal!

Basic Idea (6)



Outline of the Dual Simplex Algorithm

1. `Initialization:` Pick an optimal basis.
2. `Dual Pricing:` If all basic values are ≥ 0,
   then return OPTIMAL.
   Else pick a basic variable with value < 0.
3. `Dual Ratio test:` Compute best value preserving optimality, i.e. sign constraints on reduced costs.
   If best value does not exist,
   then return INFEASIBLE.
   Else select non-basic variable to be exchanged with violating basic variable.
4. `Update:` Update the tableau and go to 2.

Duality (1)

- The way the dual simplex works is best understood using the theory of duality
- We can get lower bounds on LP optimum value by combining constraints with convenient multipliers

$$\begin{cases} \min\ -x-y \\ 2x+y\geq 3 \\ 2x+y\leq 6 \\ x+2y\leq 6 \\ x\geq 0 \\ y\geq 0 \end{cases} \Rightarrow \begin{cases} \min\ -x-y \\ 2x+y\geq 3 \\ -2x-y\geq -6 \\ -x-2y\geq -6 \\ x\geq 0 \\ y\geq 0 \end{cases}$$

$$\begin{array}{rrcrl} 1\cdot(& -x-2y & \geq & -6 &) \\ 1\cdot(& y & \geq & 0 &) \\ \hline & -x-2y & \geq & -6 & \\ & y & \geq & 0 & \\ \hline & -x-y & \geq & -6 & \end{array}$$

Duality (2)

$$\begin{cases} \min \ -x - y \\ 2x + y \geq 3 \\ -2x - y \geq -6 \\ -x - 2y \geq -6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

$$\begin{array}{rrcrr} 1 \cdot (& 2x + y & \geq & 3 &) \\ 2 \cdot (& -2x - y & \geq & -6 &) \\ 1 \cdot (& x & \geq & 0 &) \\ \hline & 2x + y & \geq & 3 & \\ & -4x - 2y & \geq & -12 & \\ & x & \geq & 0 & \\ \hline & -x - y & \geq & -9 & \end{array}$$

Duality (3)

$$\begin{cases} \min\ -x-y \\ 2x+y \geq 3 \\ -2x-y \geq -6 \\ -x-2y \geq -6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

$$\begin{array}{rrcll} \mu_1 \cdot (& 2x+y & \geq & 3 &) \\ \mu_2 \cdot (& -2x-y & \geq & -6 &) \\ \mu_3 \cdot (& -x-2y & \geq & -6 &) \\ \hline & 2\mu_1 x + \mu_1 y & \geq & 3\mu_1 & \\ & -2\mu_2 x - \mu_2 y & \geq & -6\mu_2 & \\ & -\mu_3 x - 2\mu_3 y & \geq & -6\mu_3 & \\ \hline \end{array}$$

$$(2\mu_1 - 2\mu_2 - \mu_3)x +$$
$$(\mu_1 - \mu_2 - 2\mu_3)y \geq$$
$$3\mu_1 - 6\mu_2 - 6\mu_3$$

- If $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, $2\mu_1 - 2\mu_2 - \mu_3 \leq -1$ and $\mu_1 - \mu_2 - 2\mu_3 \leq -1$ then $3\mu_1 - 6\mu_2 - 6\mu_3$ is a lower bound

Duality (4)

- Best possible lower bound can be found by solving

$$\begin{cases} \max & 3\mu_1 - 6\mu_2 - 6\mu_3 \\ & 2\mu_1 - 2\mu_2 - \mu_3 \leq -1 \\ & \mu_1 - \mu_2 - 2\mu_3 \leq -1 \\ & \mu_1, \mu_2, \mu_3 \geq 0 \end{cases}$$

- Best solution is given by $(\mu_1, \mu_2, \mu_3) = (0, \frac{1}{3}, \frac{1}{3})$

$$\begin{array}{ll} 0 \cdot (& 2x + y \geq 3 \quad) \\ \frac{1}{3} \cdot (& -2x - y \geq -6 \quad) \\ \frac{1}{3} \cdot (& -x - 2y \geq -6 \quad) \\ \hline & -x - y \geq -4 \end{array}$$

Matches with optimum!

Dual Problem (1)

- Given a LP (called primal)

$$\min \ c^T x$$
$$Ax \geq b$$
$$x \geq 0$$

its dual is the LP

$$\max \ b^T y$$
$$A^T y \leq c$$
$$y \geq 0$$

- Primal variables associated with columns of A
- Dual variables (multipliers) associated with rows of A
- Objective and right-hand side vectors swap their roles

Dual Problem (2)

- **Prop.** The dual of the dual is the primal.

  Proof:

$$\begin{array}{lcl} \max\ b^T y & & -\min\ (-b)^T y \\ A^T y \le c & \Longrightarrow & -A^T y \ge -c \\ y \ge 0 & & y \ge 0 \end{array}$$

$$\begin{array}{lcl} -\max\ -c^T x & & \min\ c^T x \\ (-A^T)^T x \le -b & \Longrightarrow & Ax \ge b \\ x \ge 0 & & x \ge 0 \end{array}$$

- One says the primal and the dual form primal-dual pair

Dual Problem (3)

- **Prop.** $\begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array}$ and $\begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array}$ form a primal-dual pair

Proof:

$$\begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \min\ c^T x \\ Ax \geq b \\ -Ax \geq -b \\ x \geq 0 \end{array}$$

$$\begin{array}{l} \max\ b^T y_1 - b^T y_2 \\ A^T y_1 - A^T y_2 \leq c \\ y_1, y_2 \geq 0 \end{array} \quad \overset{y := y_1 - y_2}{\Longrightarrow} \quad \begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array}$$

Duality Theorems (1)

- **Th. (Weak Duality)** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{array}{l} \min \; c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max \; b^T y \\ A^T y \leq c \end{array}$$

If x is feasible solution to P and y is feasible solution to D then $y^T b \leq c^T x$

Proof:

$c - A^T y \geq 0$ and $x \geq 0$ imply $(c - A^T y)^T x \geq 0$. Hence

$$y^T b = y^T Ax = (A^T y)^T x \leq c^T x$$

Duality Theorems (2)

- Feasible solutions to D give lower bounds on P
- Feasible solutions to P give upper bounds on D
- Can the two bounds ever be equal?
- **Th. (Strong Duality)** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{array}{l} \min \; c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max \; b^T y \\ A^T y \leq c \end{array}$$

If any of P or D has a feasible solution and a finite optimum then the same holds for the other problem and the two optimum values are equal.

Duality Theorems (3)

- *Proof (Th. of Strong Duality):*
  By symmetry it is sufficient to prove only one direction. Wlog. let us assume P is feasible with finite optimum.

  After executing the Simplex algorithm to P we find B optimal feasible basis. Then:

  - $c_{\mathcal{B}}^T B^{-1} a_j = c_j$ for all $j \in \mathcal{B}$
  - $c_{\mathcal{B}}^T B^{-1} a_j \leq c_j$ for all $j \in \mathcal{R}$ (optimality conds hold)

  So $\pi^T := c_{\mathcal{B}}^T B^{-1}$ is dual feasible: $\pi^T A \leq c^T$, i.e. $A^T \pi \leq c$.
  Moreover, $c_{\mathcal{B}}^T \beta = c_{\mathcal{B}}^T B^{-1} b = \pi^T b = b^T \pi$

  By the theorem of weak duality, π is optimum for D

- If B optimal feasible basis for P, then simplex multipliers $\pi^T := c_{\mathcal{B}}^T B^{-1}$ are optimal feasible solution for D.

Duality Theorems (4)

- **Prop.** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max\ b^T y \\ A^T y \leq c \end{array}$$

If P (resp., D) has a feasible solution but the objective value is not bounded, then D (resp., P) is infeasible

Proof: By contradiction.
If y were a feasible solution to D, by weak duality theorem objective of P would be bounded from below!

Duality Theorems (5)

- **Prop.** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{aligned} &\min \; c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned} \qquad \text{and} \qquad (D) \quad \begin{aligned} &\max \; b^T y \\ &A^T y \leq c \end{aligned}$$

  If P (resp., D) has a feasible solution but the objective value is not bounded, then D (resp., P) is infeasible

- And the converse?

Duality Theorems (5)

- **Prop.** Let (P, D) be a primal-dual pair

$$(P) \quad \begin{array}{l} \min \; c^T x \\ Ax = b \\ x \geq 0 \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{l} \max \; b^T y \\ A^T y \leq c \end{array}$$

If P (resp., D) has a feasible solution but the objective value is not bounded, then D (resp., P) is infeasible

- And the converse?

$$\begin{array}{rrl} \min & 3x_1 + 5x_2 & \\ & x_1 + 2x_2 & = 3 \\ & 2x_1 + 4x_2 & = 1 \\ & x_1, x_2 \text{ free} & \end{array} \qquad \begin{array}{rrl} \max & 3y_1 + y_2 & \\ & y_1 + 2y_2 & = 3 \\ & 2y_1 + 4y_2 & = 5 \\ & x_1, x_2 \text{ free} & \end{array}$$

Duality Theorems (6)

$$
\begin{array}{lcl}
\text{Primal unbounded} & \Longrightarrow & \text{Dual infeasible} \\
\text{Dual unbounded} & \Longrightarrow & \text{Primal infeasible} \\
\text{Primal infeasible} & \Longrightarrow & \text{Dual} \begin{cases} \text{infeasible} \\ \text{unbounded} \end{cases} \\
\text{Dual infeasible} & \Longrightarrow & \text{Primal} \begin{cases} \text{infeasible} \\ \text{unbounded} \end{cases}
\end{array}
$$

Karush Kuhn Tucker Optimality Conds (1)

- Consider a primal-dual pair of the form

$$\begin{array}{lcl} \min\ c^T x & & \max\ b^T y \\ Ax = b & \text{and} & A^T y + w = c \\ x \geq 0 & & w \geq 0 \end{array}$$

- Karush-Kuhn-Tucker (KKT) optimality conditions are
  - $Ax = b$
  - $A^T y + w = c$
  - $x, w \geq 0$
  - $x^T w = 0$ (complementary slackness)
- They are necessary and sufficient conditions for optimality of the pair of primal-dual solutions $(x, (y, w))$
- Used, e.g., as a test for quality in LP solvers

Karush Kuhn Tucker Optimality Conds (2)

$$\begin{array}{llll} & \min\ c^T x & & \max\ b^T y \\ (P) & Ax = b & (D) & A^T y + w = c \\ & x \geq 0 & & w \geq 0 \end{array}$$

(KKT)

- $Ax = b$
- $A^T y + w = c$
- $x, w \geq 0$
- $x^T w = 0$

- **Th.** $(x, (y, w))$ is solution to KKT iff
  x optimal solution to P and (y, w) optimal solution to D

  Proof:

  $\Rightarrow$ By $0 = x^T w = x^T (c - A^T y) = c^T x - b^T y$, Weak Duality

  $\Leftarrow$ x is feasible solution to P, (y, w) is feasible solution to D. By Strong Duality $x^T w = x^T (c - A^T y) = c^T x - b^T y = 0$ as both solutions are optimal

Towards Dual Simplex: Relating Bases (1)

$$(P)\quad \begin{aligned} \min\ \ & z = c^T x \\ & Ax = b \\ & x \geq 0 \end{aligned} \qquad (D)\quad \begin{aligned} \max\ \ & Z = b^T y \\ & A^T y \leq c \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} \max\ \ & Z = b^T y \\ & A^T y + w = c \\ & w \geq 0 \end{aligned}$$

- Let $\mathcal{B}$ be basis of P.
  Reorder rows in D so that $\mathcal{B}$-basic variables are first m.
  Reorder columns in D so that the matrix is

$$\left(\begin{array}{c|c|c} B^T & I & 0 \\ \hline R^T & 0 & I \end{array} \right) \left(\begin{array}{c} y \\ w_{\mathcal{B}} \\ w_{\mathcal{R}} \end{array} \right)$$

  Submatrix of vars y and vars $w_{\mathcal{R}}$:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array} \right)$$

Towards Dual Simplex: Relating Bases (2)

$\hat{\mathcal{B}} = (y, w_{\mathcal{R}})$ is a basis of D:

$$\hat{B} = \left(\begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array}\right)$$

$$\hat{B}^{-1} = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right)$$

- Each var w_j in D is associated to var a x_j in P.
- w_j is $\hat{\mathcal{B}}$-basic iff x_j is not $\mathcal{B}$-basic

Dual Feasibility is Primal Optimality

- Let's apply simplex algorithm to dual problem
- Let's see correspondence of dual feasibility in primal LP

$$\hat{B}^{-1}c = \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array}\right)\left(\begin{array}{c} c_{\mathcal{B}} \\ \hline c_{\mathcal{R}} \end{array}\right) = \left(\begin{array}{c} B^{-T} c_{\mathcal{B}} \\ \hline -R^T B^{-T} c_{\mathcal{B}} + c_{\mathcal{R}} \end{array}\right)$$

- There is no restriction on the sign of $y_1, \ldots, y_m$
- Variables w_j have to be non-negative. But

$$-R^T B^{-T} c_{\mathcal{B}} + c_{\mathcal{R}} \geq 0 \quad \text{iff} \quad c_{\mathcal{R}}^T - c_{\mathcal{B}}^T B^{-1} R \geq 0 \quad \text{iff} \quad d_{\mathcal{R}}^T \geq 0$$

- $\hat{\mathcal{B}}$ is dual feasible iff $d_j \geq 0$ for all $j \in \mathcal{R}$
- Dual feasibility is primal optimality!

Dual Optimality is Primal Feasibility

- $\hat{\mathcal{B}}$-basic dual vars: $(y \mid w_{\mathcal{R}})$ with costs $(b^T \mid 0)$
- Non $\hat{\mathcal{B}}$-basic dual vars: $w_{\mathcal{B}}$ with costs (0)
- Optimality condition: reduced costs ≤ 0 (maximization!)

$$0 \geq \left(\; 0 \; \right) - \left(\; b^T \; \middle| \; 0 \; \right) \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array} \right) \left(\begin{array}{c} I \\ \hline 0 \end{array} \right) =$$

$$\left(\; 0 \; \right) - \left(\; b^T B^{-T} \; \middle| \; 0 \; \right) \left(\begin{array}{c} I \\ \hline 0 \end{array} \right) = \left(\; -\beta^T \; \right) \quad \text{iff} \quad \beta \geq 0$$

- For all $1 \leq p \leq m$, w_{k_p} is not dual improving iff $\beta_p \geq 0$
- Dual optimality is primal feasibility!

Improving a Non-Optimal Solution (1)

- Let p ($1 \leq p \leq m$) be such that $\beta_p < 0 \Leftrightarrow b^T B^{-T} e_p < 0$
  Let $\rho_p = B^{-T} e_p$, so $b^T \rho_p = \beta_p$. If w_{k_p} takes value $t \geq 0$:

$$\begin{pmatrix} y(t) \\ w_{\mathcal{R}}(t) \end{pmatrix} = \hat{B}^{-1} c - \hat{B}^{-1} t e_p =$$

$$\begin{pmatrix} B^{-T} c_{\mathcal{B}} \\ d_{\mathcal{R}} \end{pmatrix} - \left(\begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array} \right) \left(\begin{array}{c} t e_p \\ \hline 0 \end{array} \right) =$$

$$\left(\begin{array}{c} B^{-T} c_{\mathcal{B}} - t\rho_p \\ \hline d_{\mathcal{R}} + tR^T \rho_p \end{array} \right)$$

- Dual objective value improvement is

$$\Delta Z = b^T y(t) - b^T y(0) = -t b^T \rho_p = -t\beta_p$$

Improving a Non-Optimal Solution (2)

- Only w variables need to be ≥ 0: for $j \in \mathcal{R}$

$$w_j(t) = d_j + t a_j^T \rho_p = d_j + t \rho_p^T a_j = \\ d_j + t e_p^T B^{-1} a_j = d_j + t e_p^T \alpha_j = d_j + t \alpha_j^p$$

$$w_j(t) \geq 0 \iff d_j + t \alpha_j^p \geq 0$$

  - If $\alpha_j^p \geq 0$ the constraint is satisfied for all $t \geq 0$
  - If $\alpha_j^p < 0$ we need $\frac{d_j}{-\alpha_j^p} \geq t$
- Best improvement achieved with

$$\Theta_D := \min\{\frac{d_j}{-\alpha_j^p} \mid \alpha_j^p < 0\}$$

- Variable w_q is blocking when $\Theta_D = \frac{d_q}{-\alpha_q^p}$

Improving a Non-Optimal Solution (3)

1. If $\Theta_D = +\infty$ (there is no j such that $j \in \mathcal{R}$ and $\alpha_j^p < 0$):

   Value of dual objective can be increased infinitely.
   Dual LP is unbounded.
   Primal LP is infeasible.

2. If $\Theta_D < +\infty$ and w_q is blocking:

   When setting $w_{k_p} = \Theta_D$ sign of dual slack basic vars (primal reduced costs of non-basic vars) is respected

   In particular, $w_q(\Theta_D) = d_q + \Theta_D \alpha_q^p = d_q + (\frac{d_q}{-\alpha_q^p})\alpha_q^p = 0$

   We can make a basis change:

   - In dual: w_{k_p} enters $\hat{\mathcal{B}}$ and w_q leaves
   - In primal: x_{k_p} leaves $\mathcal{B}$ and x_q enters

Update

- We forget about dual LP and work only with primal LP
- New basic indices: $\bar{\mathcal{B}} = \mathcal{B} - \{k_p\} \cup \{q\}$
- New objective value: $\bar{Z} = Z - \Theta_D \beta_p$
- New dual basic sol: $\bar{y} = y - \Theta_D \rho_p$
  $\bar{d}_j = d_j + \Theta_D \alpha_j^p$ if $j \in \mathcal{R}$, $\bar{d}_{k_p} = \Theta_D$
- New primal basic sol: $\bar{\beta}_p = \Theta_P$, $\bar{\beta}_i = \beta_i - \Theta_P \alpha_q^i$ if $i \neq p$
  where $\Theta_P = \frac{\beta_p}{\alpha_q^p}$
- New basis inverse: $\bar{B}^{-1} = EB^{-1}$
  where $E = (e_1, \ldots, e_{p-1}, \eta, e_{p+1}, \ldots, e_m)$ and
  $\eta^T = \left(\left(\frac{-\alpha_q^1}{\alpha_q^p}\right), \ldots, \left(\frac{-\alpha_q^{p-1}}{\alpha_q^p}\right), \frac{1}{\alpha_q^p} \left(\frac{-\alpha_q^{p+1}}{\alpha_q^p}\right), \ldots, \left(\frac{-\alpha_q^m}{\alpha_q^p}\right) \right)^T$

Algorithmic Description (1)

1. `Initialization:` Find an initial dual feasible basis $\mathcal{B}$
   Compute B^{-1}, $\beta = B^{-1}b$,
   $y^T = c_{\mathcal{B}}^T B^{-1}$, $d_{\mathcal{R}}^T = c_{\mathcal{R}}^T - y^T R$, $Z = b^T y$

2. `Dual Pricing:`
   If for all $i \in \mathcal{B}, \beta_i \geq 0$ then return OPTIMAL
   Else let p be such that $\beta_p < 0$.
   Compute $\rho_p^T = e_p^T B^{-1}$ and $\alpha_j^p = \rho_p^T a_j$ for $j \in \mathcal{R}$

3. `Dual Ratio test:` Compute $\mathcal{J} = \{j \mid j \in \mathcal{R}, \alpha_j^p < 0\}$.
   If $\mathcal{J} = \emptyset$ then return INFEASIBLE
   Else compute $\Theta_D = \min_{j \in \mathcal{J}}(\frac{d_j}{-\alpha_j^p})$ and q st. $\Theta_D = \frac{d_q}{-\alpha_q^p}$

Algorithmic Description (2)

4. `Update:`

   $\bar{\mathcal{B}} = \mathcal{B} - \{k_p\} \cup \{q\}$
   $\bar{Z} = Z - \Theta_D \beta_p$

   <u>`Dual solution`</u>

   $\bar{y} = y - \Theta_D \rho_p$
   $\bar{d}_j = d_j + \Theta_D \alpha_j^p$ if $j \in \mathcal{R}$, $\bar{d}_{k_p} = \Theta_D$

   <u>`Primal solution`</u>

   Compute $\alpha_q = B^{-1} a_q$ and $\Theta_P = \frac{\beta_p}{\alpha_q^p}$
   $\bar{\beta}_p = \Theta_P$, $\bar{\beta}_i = \beta_i - \Theta_P \alpha_q^i$ if $i \neq p$

   $\bar{B}^{-1} = E B^{-1}$

   Go to 2.

Primal vs. Dual Simplex

PRIMAL	DUAL
• Ratio test: $\mathcal{O}(m)$ divs	• Ratio test: $\mathcal{O}(n-m)$ divs
• Can handle bounds efficiently	• Can handle bounds efficiently (not explained here)
• Many years of research and implementation	• Developments in the 90's made it an alternative
• There are classes of LP's for which it is the best	• Nowadays on average it gives better performance
• Not suitable for solving LP's with integer variables	• Suitable for solving LP's with integer variables