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Generalized Jacobi and Gauss-Seidel Methods for Solving Linear System of Equations

Davod Khojasteh Salkuyeh*
(*Department of Mathematics, Mohaghegh Ardabili University, P.O. Box 56199-11367, Ardabil, Iran*
E-mail: khojaste@uma.ac.ir)

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Abstract

The Jacobi and Gauss-Seidel algorithms are among the stationary iterative methods for solving linear system of equations. They are now mostly used as preconditioners for the popular iterative solvers. In this paper a generalization of these methods are proposed and their convergence properties are studied. Some numerical experiments are given to show the efficiency of the new methods.

Keywords: Jacobi; Gauss-Seidel; generalized; convergence.
Mathematics subject classification: 65F10, 65F50

1. Introduction

Consider the linear system of equations

$$Ax = b, \tag{1.1}$$

where the matrix $A \in \mathbb{R}^{n\times n}$ and $x, b \in \mathbb{R}^n$. Let A be a nonsingular matrix with nonzero diagonal entries and

$$A = D - E - F,$$

where D is the diagonal of A, $-E$ its strict lower part, and $-F$ its strict upper part. Then the Jacobi and the Gauss-Seidel methods for solving Eq. (1.1) are defined as

$$x_{k+1} = D^{-1}(E+F)x_k + D^{-1}b,$$
$$x_{k+1} = (D-E)^{-1}Fx_k + (D-E)^{-1}b,$$

respectively. There are many iterative methods such as GMRES [7] and Bi-CGSTAB [9] algorithms for solving Eq. (1.1) which are more efficient than the Jacobi and Gauss-Seidel methods. However, when these methods are combined with the more efficient methods, for example as a preconditioner, can be quite successful. For example see [4, 6]. It has been proved that if A is a strictly diagonally dominant (SDD) or irreducibly diagonally dominant, then the associated Jacobi and Gauss-Seidel iterations converge for any initial guess x_0 [6]. If A is a symmetric positive definite (SPD) matrix, then the Gauss-Seidel

*Corresponding author.

method also converges for any x_0 [1]. In this paper we generalize these two methods and study their convergence properties.

This paper is organized as follows. In Section 2, we introduce the new algorithms and verify their properties. Section 3 is devoted to the numerical experiments. In Section 4 some concluding remarks are also given.

2. Generalized Jacobi and Gauss-Seidel methods

Let $A = (a_{ij})$ be an $n \times n$ matrix and $T_m = (t_{ij})$ be a banded matrix of bandwidth $2m+1$ defined as

$$t_{ij} = \begin{cases} a_{ij}, & \mid i-j \mid \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the decomposition $A = T_m - E_m - F_m$ where $-E_m$ and $-F_m$ are the strict lower and upper part of the matrix $A_m - T_m$, respectively. In other words matrices T_m, E_m and F_m are defined as following

$$T_m = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m+1} & & \\ \vdots & \ddots & & \ddots & \\ a_{m+1,1} & & \ddots & & a_{n-m,n} \\ & \ddots & & \ddots & \vdots \\ & & a_{n,n-m} & \cdots & a_{n,n} \end{pmatrix},$$

$$E_m = \begin{pmatrix} & & & \\ & & & \\ -a_{m+2,1} & & & \\ \vdots & \ddots & & \\ -a_{n,1} & \cdots & -a_{n-m-1,n} & \end{pmatrix},$$

$$F_m = \begin{pmatrix} & -a_{1,m+2} & \cdots & -a_{1,n} \\ & & \ddots & \vdots \\ & & & -a_{n-m-1,n} \\ & & & \\ & & & \end{pmatrix}.$$

Then we define the generalized Jacobi (GJ) and generalized Gauss-Seidel (GGS) iterative methods as follows

$$x_{k+1} = T_m^{-1}(E_m + F_m)x_k + T_m^{-1}b, \tag{2.1}$$

$$x_{k+1} = (T_m - E_m)^{-1}F_m x_k + (T_m - E_m)^{-1}b, \tag{2.2}$$

respectively. Let $B_{GJ}^{(m)} = T_m^{-1}(E_m+F_m)$, and $B_{GGS}^{(m)} = (T_m-E_m)^{-1}F_m$, be the iteration matrices of the GJ and GGS methods, respectively. Note that if $m = 0$ then (2.1) and (2.2) result in the Jacobi and Gauss-Seidel methods. We now study the convergence of the new methods. To do so, we introduce the following definition.

Definition 2.1. *An $n \times n$ matrix $A = (a_{ij})$ is said to be strictly diagonally dominant (SDD) if*

$$| a_{ii} | > \sum_{j=1, j\neq i}^{n} | a_{ij} |, \quad i = 1, \cdots, n.$$

Theorem 2.1. *Let A be an SDD matrix. Then for any natural number $m \leq n$ the GJ and GGS methods are convergent for any initial guess x_0.*

Proof. Let $M = (M_{ij})$ and $N = (N_{ij})$ be $n \times n$ matrices with M being SDD. Then (see [2], Lemma 1)

$$\rho(M^{-1}N) \leq \rho = \max_i \rho_i, \tag{2.3}$$

where

$$\rho_i = \frac{\sum_{j=1}^{n} | N_{ij} |}{| M_{ii} | - \sum_{j=1,\ j\neq i}^{n} | M_{ij} |}.$$

Now, let $M = T_m$ and $N = E_m + F_m$ in the GJ method and $M = T_m - E_m$ and $N = F_m$ in the GGS method. Obviously, in the both cases the matrix M is SDD. Hence M and N satisfy relation (2.3). Having in mind that the matrix A is an SDD matrix, it can be easily verified that $\rho_i < 1$. Therefore $\rho(M^{-1}N) \leq \rho < 1$ and this completes the proof. ■

The definition of matrix M and N in Theorem 2.1 depend on the parameter m. Hence, for later use, we denote ρ by $\rho^{(m)}$. By a little computation one can see that

$$\rho^{(1)} \geq \rho^{(2)} \geq \cdots \geq \rho^{(n)} = 0. \tag{2.4}$$

By this relation we can not deduce that

$$\rho(B_{GJ}^{(m+1)}) \leq \rho(B_{GJ}^{(m)}),$$

or

$$\rho(B_{GGS}^{(m+1)}) \leq \rho(B_{GGS}^{(m)}).$$

But Eq. (2.4) shows that we can choose a natural number $m \leq n$ such that $\rho(B_{GJ}^{(m)})$ and $\rho(B_{GGS}^{(m)})$ are sufficiently small. To illustrate this, consider the matrix

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & -1 & 0 \\ 1 & 1 & -4 & 1 \\ -1 & -1 & -1 & 4 \end{pmatrix}.$$

Obviously, A is an SDD matrix. Here we have $\rho(B_J) = 0.3644 < \rho(B_{GJ}^{(1)}) = 0.4048$, where B_J is the iteration matrix of the Jacobi method. On the other hand we have

$$\rho(B_J) = 0.3644 > \rho(B_{GJ}^{(2)}) = 0.2655.$$

For the GGS method the result is very suitable since

$$\rho(B_G) = 0.2603 > \rho(B_{GGS}^{(1)}) = 0.1111 > \rho(B_{GGS}^{(2)}) = 0.0968,$$

where B_G is the iteration matrix of the Gauss-Seidel method.

Now we study the new methods for an another class of matrices. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices. Then $A \leq B$ $(A < B)$ if by definition,

$$a_{ij} \leq b_{ij} \ (a_{ij} < b_{ij}) \quad \text{for} \ \ 1 \leq i, \ \ j \leq n.$$

For $n \times n$ real matrices A, M, and N, $A = M - N$ is said to be a regular splitting of the matrix A if M is nonsingular with $M^{-1} \geq O$ and $N \geq O$, where O is the $n \times n$ zero matrix. A matrix $A = (a_{ij})$ is said to be an M-matrix (MP-matrix) if $a_{ii} > 0$ for $i = 1, \cdots, n$, $a_{ij} \leq 0$, for $i \neq j$, A is nonsingular and $A^{-1} \geq O (A^{-1} > O)$.

Theorem 2.2. (Saad [6]) *Let $A = M - N$ be a regular splitting of the matrix A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and $A^{-1} \geq O$.*

Theorem 2.3. (Saad [6]) *Let $A = (a_{ij})$, $B = (b_{ij})$ be two matrices such that $A \leq B$ and $b_{ij} \leq 0$ for all $i \neq j$. Then if A is an M-matrix, so is the matrix B.*

Theorem 2.4. *Let A be an M-matrix. Then for a given natural number $m \leq n$, both of the GJ and GGS methods are convergent for any initial guess x_0.*

Proof. Let $M_m = T_m$ and $N_m = E_m + F_m$ in the GJ method and $M_m = T_m - E_m$ and $N_m = F_m$ in the GGS method. Obviously, in the both cases we have $A \leq M_m$. Hence by Theorem 2.3, we conclude that the matrix M_m is an M-matrix. On the other hand we have $N_m \geq O$. Therefore, $A = M_m - N_m$ is a regular splitting of the matrix A. Having in mind that $A^{-1} \geq 0$ and Theorem 2.2 we deduce that $\rho(B_{GJ}^{(m)}) < 1$ and $\rho(B_{GGS}^{(m)}) < 1$. ∎

Theorem 2.5. (Varga [8]) *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splitting of A, where $A^{-1} > O$. If $N_2 \geq N_1 \geq O$ such that neither N_1 nor $N_2 - N_1$ is the null matrix, then*

$$0 < \rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2) < 1.$$

For the MP-matrices, the next theorem shows that larger m results in smaller spectral radius of the iteration matrix of GJ and GGS iterations.

Theorem 2.6. *Let A be an MP-matrix, p and q be two natural numbers such that $0 \leq p < q \leq n$ and for a given natural number $m \leq n$, M_m and N_m be the matrices introduced in the proof of Theorem 2.4 for the GJ and GGS methods. Moreover let neither N_p nor $N_p - N_q$ is the null matrix. Then*

$$\rho(B_{GJ}^{(q)}) < \rho(B_{GJ}^{(p)}), \quad \rho(B_{GGS}^{(q)}) < \rho(B_{GGS}^{(p)}).$$

Table 3.1: Numerical results for $g(x,y)=\exp(xy)$. Timings are in second.

nx	Jacobi	GJ	Gauss-Seidel	GGS
20	1169(0.11)	526(0.08)	613(0.06)	60(0.02)
30	2511(0.69)	1088(0.31)	1318(0.25)	63(0.02)
40	4335(1.45)	1825(1.02)	2227(0.77)	65(0.05)

Table 3.2: Numerical results for $g(x,y)=x+y$. Timings are in second.

nx	Jacobi	GJ	Gauss-Seidel	GGS
20	1184(0.11)	533(0.06)	621(0.06)	60(0.02)
30	2544(0.56)	1102(0.33)	1136(0.27)	63(0.03)
40	4392(1.47)	1849(1.02)	2307(0.88)	65(0.05)

Table 3.3: Numerical results for $g(x,y)=0$. Timings are in second.

nx	Jacobi	GJ	Gauss-Seidel	GGS
20	1236(0.14)	556(0.08)	649(0.08)	60(0.02)
30	2658(0.53)	1150(0.34)	1395(0.28)	63(0.03)
40	4591(1.56)	1931(1.08)	2412(0.95)	65(0.05)

Table 3.4: Numerical results for $g(x,y)=-\exp(4xy)$. Timings are in second.

nx	Jacobi	GJ	Gauss-Seidel	GGS
80	†	7869(18.5)	†	68(0.16)
90	†	9718(29.32)	†	68(0.20)
100	†	†	†	68(0.27)

Proof. By Theorem 2.4 we see that $A=M_p-N_p=M_q-N_q$ are two regular splitting of the matrix A. It can be easily seen that $O\le N_q\le N_p$. A is an MP-matrix hence $A^{-1}>O$. Therefore all the hypothesis of Theorem 2.5 were provided. Hence the desired result is obtained. ■

3. Numerical examples

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer Pentium 4 - 256 MHz. For the numerical experiments we consider the equation

$$-\Delta u+g(x,y)u=f(x,y),\qquad (x,y)\in\Omega=(0,1)\times(0,1). \tag{3.1}$$

Discretizing Eq. (3.1) on an $nx\times ny$ grid, by using the second order centered differences for the Laplacian gives a linear system of equations of order $n=nx\times ny$ with n unknowns $u_{ij}=u(ih,jh)$ $(1\le i,j\le n)$:

$$-u_{i-1,j}-u_{i,j-1}+(4+h^2g(ih,jh))u_{ij}-u_{i+1,j}-u_{i,j+1}=h^2f(ih,jh).$$

The boundary conditions are taken so that the exact solution of the system is $x = [1, \cdots, 1]^T$. Let $nx = ny$. We consider the linear systems arisen from this kind of discretization for three functions $g(x, y) = \exp(xy)$, $g(x, y) = x + y$ and $g(x, y) = 0$.

It can be easily verified that the coefficient matrices of these systems are M-matrix (see for more details [3, 5]). For each function we give the numerical results of the methods described in section 2 for $nx = 20, 30, 40$. The stopping criterion $\| x_{k+1} - x_k \|_2 < 10^{-7}$, was used and the initial guess was taken to be zero vector. For the GJ and GGS methods we let $m = 1$. Hence T_m is a tridiagonal matrix. In the implementation of the GJ and GGS methods we used the LU factorization of T_m and $T_m - E_m$, respectively. Numerical results are given in Tables 3.1, 3.2, and 3.3. In each table the number of iterations of the method and the CPU time (in parenthesis) for convergence are given. We also give the numerical results related to the function $g(x, y) = -\exp(4xy)$ in Table 3.4 for $nx = 80, 90, 100$. In this table a (†) shows that we have not the solution after 10000 iterations. As the numerical results show the GJ and GGS methods are more effective than the Jacobi and Gauss-Seidel methods, respectively.

4. Conclusion

In this paper, we have proposed a generalization of the Jacobi and Gauss-Seidel methods say GJ and GGS, respectively, and studied their convergence properties. In the decomposition of the coefficient matrix a banded matrix T_m of bandwidth $2m + 1$ is chosen. Matrix T_m is chosen such that the computation of $w = T_m^{-1} y$ (in GJ method) and $w = (T_m - E_m)^{-1} y$ (in GGS method) can be easily done for any vector y. To do so one may use the LU factorization of T_m and $T_m - E_m$. In practice m is chosen very small, e.g., $m = 1, 2$. For $m = 1$, T_m is a tridiagonal matrix and its LU factorization can be easily obtained. The new methods are suitable for sparse matrices such as matrices arisen from discretization of the PDEs. These kinds of matrices are usually pentadiagonal. In this case for $m = 1$, T_m is tridiagonal and each of the matrices E_m and F_m contains only one nonzero diagonal and a few additional computations are needed in comparing with Jacobi and Gauss-Seidel methods (as we did in this paper). Numerical results show that the new methods are more effective than the conventional Jacobi and Gauss-Seidel methods.

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