Integer Programming

Wolfram Wiesemann

November 17, 2009

Contents of this Lecture

Contents of this Lecture

Definitions

- **Mixed Integer Programming Problem.**

$$\min \; x_0 = c^{\mathrm{T}} x$$

subject to

$$\begin{aligned} Ax &= b & & \\ x_j &\geq 0 & & \text{for } j \in N = \{1, \dots, n\} \\ x_j &\in \mathbb{Z} & & \text{for } j \in Z \subseteq N. \end{aligned}$$

Note: $x_j \in N \setminus Z$ are continuous, as before.

- **Pure Integer Programming Problem.** $Z = N \cup \{x_0\}$, i.e., all variables (including slack and objective value) are integral. Can be achieved by *scaling*.

Contents of this Lecture

Obtaining a Pure Integer Progamming Problem

Consider the following problem:

$$\min\ x_0 = -\frac{1}{3}x_1 - \frac{1}{2}x_2$$

subject to

$$\begin{aligned}
\frac{2}{3}x_1 + \frac{1}{3}x_2 &\leq \frac{4}{3} \\
\frac{1}{2}x_1 - \frac{3}{2}x_2 &\leq \frac{2}{3} \\
x_1, x_2 &\geq 0 \\
x_1, x_2 &\in \mathbb{Z}.
\end{aligned}$$

This is not a pure integer programming problem:

- x_0 not integral.
- Slack variables not integral.

Obtaining a Pure Integer Progamming Problem

Step 1. Scale the equations of the model.

$$\min \ x_0' = -2x_1 - 3x_2 \qquad (*6)$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 4 \qquad (*3) \\ 3x_1 - 9x_2 &\leq 4 \qquad (*6) \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{Z}. \end{aligned}$$

Obtaining a Pure Integer Progamming Problem

Step 2. Insert (integral) slack variables:

$$\min \ x_0' = -2x_1 - 3x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 + {\color{red}x_3} &= 4 \\ 3x_1 - 9x_2 + {\color{red}x_4} &= 4 \\ x_1, x_2, {\color{red}x_3}, {\color{red}x_4} &\geq 0 \\ x_1, x_2, {\color{red}x_3}, {\color{red}x_4} &\in \mathbb{Z}. \end{aligned}$$

Contents of this Lecture

Capital Budgeting

- Company has resources $i \in \{1, \dots, m\}$. Resource i has limited availability b_i.
- Company can undertake projects $j \in \{1, \dots, n\}$. Project j requires a_{ij} units of resource i and gives revenues c_j.
- Which projects should be undertaken such that the resource availabilities are observed and the revenues maximised?

$$\max_x \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \forall\, i \in \{1, \dots, m\}$$

$$x_j \in \{0, 1\} \qquad \forall\, j \in \{1, \dots, n\}$$

Contents of this Lecture

Depot Location (1)

- Company has m potential distribution sites $i \in \{1, \ldots, m\}$. Building a distribution centre at site i costs f_i.
- There are n customers $j \in \{1, \ldots, n\}$, each of whose demands need to be satisfied from one or more distribution centres. Satisfying fraction x_{ij} of customer j's demand from distribution centre i costs c_{ij}, given that centre i is built.
- Which distribution centres should be built, and how should the customer demand's be satisfied, to minimise costs?

Depot Location (2)

$$\min_{x,y} \sum_{i=1}^{m} f_i y_i + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad \forall\, j \in \{1, \dots, n\}$$

$$x_{ij} \le y_i \qquad \forall\, i \in \{1, \dots, m\},\ j \in \{1, \dots, n\}$$

$$x_{ij} \ge 0 \qquad \forall\, i \in \{1, \dots, m\},\ j \in \{1, \dots, n\}$$

$$y_i \in \{0, 1\} \qquad \forall\, i \in \{1, \dots, m\}$$

Contents of this Lecture

Finite-Valued Variables

Assume a variable x_j can only take a finite number of values:

$$x_j \in \{p_1, \ldots, p_m\}.$$

We can introduce variables $z_j^1, \ldots, z_j^m \in \{0, 1\}$ and add the constraint

$$z_j^1 + \ldots + z_j^m = 1.$$

Now, we can substiute x_j with

$$p_1 z_j^1 + \ldots + p_m z_j^m$$

in the objective function and all constraints.

Finite-Valued Variables

Example. $x_j \in \{1, 3, 11\}$ can be modeled as

$$\begin{aligned} z_j^1 + z_j^2 + z_j^3 &= 1 \\ z_j^1, z_j^2, z_j^3 &\in \{0, 1\} \,. \end{aligned}$$

We then substitute x_j everywhere by

$$1z_j^1 + 3z_j^2 + 11z_j^3 .$$

Exercise. Is it possible to save variables here?

Contents of this Lecture

Logical Operations

We can model logical operations on the constraints via integer variables. For example, the expression

$$a^{\mathrm{T}} x \leq b \quad \vee \quad d^{\mathrm{T}} x \leq e$$

can be expressed by:

$$\begin{aligned} a^{\mathrm{T}} x - \mathrm{M}\delta &\leq b \\ d^{\mathrm{T}} x - \mathrm{M}(1-\delta) &\leq e \\ \delta &\in \{0,1\}\,, \end{aligned}$$

where M is a large number.

Logical Operations

Example. We want to model the following problem:

$$\min\ x$$

subject to

$$x \in [0,1] \quad \vee \quad x \geq 2.$$

Solution. This can be expressed as:

$$\min\ x$$

subject to

$$\begin{aligned} x &\leq 1 + \mathrm{M}\delta \\ x &\geq 2 - \mathrm{M}(1-\delta) \\ x &\geq 0. \end{aligned}$$

Logical Operations

Example. We want to model the following problem:

$$\min \; x_1 - x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\le 4 \\ x_1 \ge 1 \quad \vee \quad x_2 &\ge 1 \quad \text{but not both } x_1, x_2 > 1 \\ x_1, x_2 &\ge 0. \end{aligned}$$

Logical Operations

Solution. This can be expressed as:

$$\min \; x_1 + x_2$$

subject to

$$\begin{aligned}
x_1 + x_2 &\leq 4 \\
x_1 &\geq 1 - \mathrm{M}\delta \\
x_2 &\geq 1 - \mathrm{M}(1 - \delta) \\
x_1 &\leq 1 + \mathrm{M}(1 - \delta) \\
x_2 &\leq 1 + \mathrm{M}\delta \\
x_1, x_2 &\geq 0.
\end{aligned}$$

Contents of this Lecture

The Big Picture

Want to solve a pure IP problem:

- Don't know how to solve IP problems.
- Know how to solve continuous problems (Simplex).

Outline of a solution procedure:

- Solve a continuous *relaxation*.
  - Contains all originally feasible solutions, plus others.
- If optimal solution is integral, we are done.
- Otherwise, *tighten* the relaxation and repeat.

Continuous relaxation: $x_j \in \mathbb{Z} \rightsquigarrow x_j \in \mathbb{R}$.
Tightening: Add *cutting planes* (cut off current optimum).

The Big Picture

A *cutting plane algorithm* to solve pure integer programming problems works as follows.

1. Solve the IP problem with *continuous* variables instead of discrete ones.
2. If the resulting optimal solution x^* is integral, stop $\Rightarrow$ optimal solution found.
3. Generate a *cut*, i.e., a constraint which is satisfied by all feasible integer solutions but not by x^*.
4. Add this new constraint, resolve problem, and go back to (2).

Terminates after finite number of iterations in (2). Resulting x^* is integral and optimal.

Example

Consider the following problem:

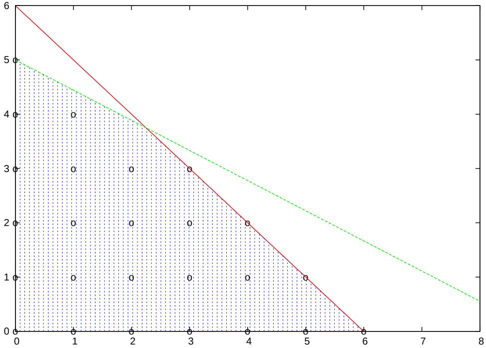
$$\max \; x_0 = 5x_1 + 8x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 6 \\ 5x_1 + 9x_2 &\leq 45 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{Z}. \end{aligned}$$

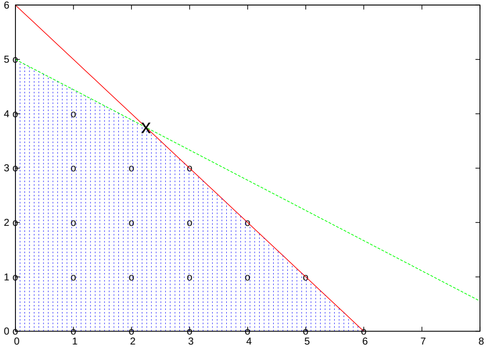
Example

Step 1. Solve the IP problem with *continuous* variables instead of discrete ones.



Example

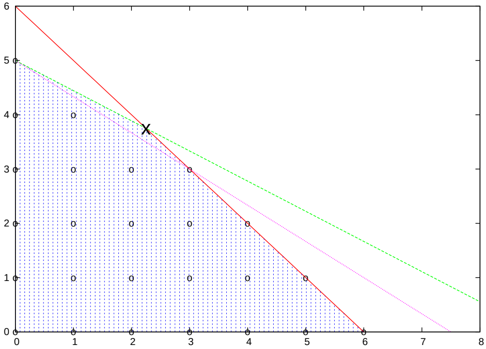
Step 2. If the resulting optimal solution x^* is integral, stop $\Rightarrow$ optimal solution found.



Resulting solution is $x^* = (2.25, 3.75)$ and hence *not* integral.

Example

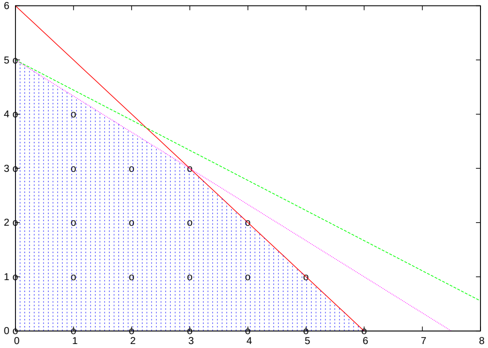
Step 3. Generate a *cut*, i.e., a constraint which is satisfied by all feasible integer solutions but not by x^*.



We generate cut $2x_1 + 3x_2 \leq 15$.

Example

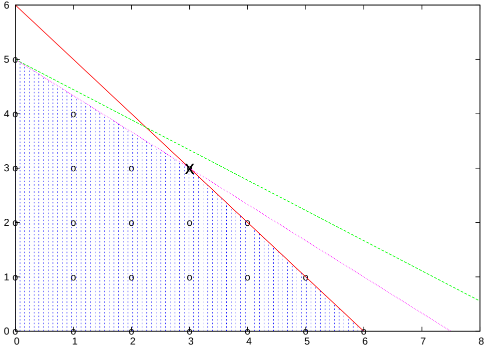
Step 4. Add this new constraint, resolve problem, and go back to (2).



New optimal solution is $x^* = (3, 3)$.
Note: previous x^* is not feasible anymore.

Example

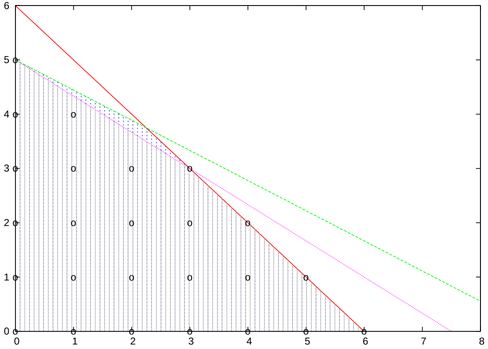
Step 2. If the resulting optimal solution x^* is integral, stop $\Rightarrow$ optimal solution found.



$x^* = (3, 3)$ is integral $\Rightarrow$ optimal solution found.

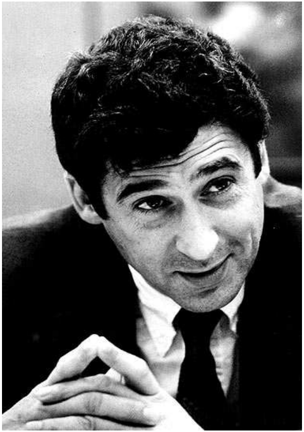
Example

Remark. The cut we introduced only removed *non-integral* solutions. Cuts *never* cut off feasible solutions of the original IP problem!



Contents of this Lecture

Ralph E. Gomory (* 1929)



"Outline of an Algorithm for Integer Solutions to Linear Programs"
Bulletin of the American Mathematical Society, Vol. 64,
pp. 275-278, **1958**.

Gomory Cut

Assume $x_1, \ldots, x_n \geq 0$ and integral. We show how to construct a Gomory Cut for

$$a_1 x_1 + \ldots + a_n x_n = b,$$

where $a_j, b \in \mathbb{R}$ (not necessarily integral). Note that this can be written as

$$(\lfloor a_1 \rfloor + \underbrace{[a_1 - \lfloor a_1 \rfloor]}_{f_1}) x_1 + \ldots + (\lfloor a_n \rfloor + \underbrace{[a_n - \lfloor a_n \rfloor]}_{f_n}) x_n = \lfloor b \rfloor + \underbrace{[b - \lfloor b \rfloor]}_{f},$$

where $\lfloor \beta \rfloor = \max \{\alpha \in \mathbb{Z} : \alpha \leq \beta\}$ (largest integer smaller than or equal to β).

Gomory Cut

We separate fractional and integral terms:

$$
\begin{aligned}
& (\lfloor a_1 \rfloor + f_1)x_1 + \ldots + (\lfloor a_n \rfloor + f_n)x_n = \lfloor b \rfloor + f \\
\Leftrightarrow \quad & f_1 x_1 + \ldots + f_n x_n - f = \lfloor b \rfloor - \lfloor a_1 \rfloor x_1 - \ldots - \lfloor a_n \rfloor x_n.
\end{aligned}
$$

Observations.

1. As $x_j \in \mathbb{Z}$ for all feasible x, right-hand side is integral.
2. Thus, for all feasible x, left-hand side must be integral, too.
3. As $0 \leq f < 1$, $x \geq 0$ and left-hand side integral, left-hand side must be non-negative.

Consequence. $f_1 x_1 + \ldots + f_n x_n - f \geq 0 \;\Leftrightarrow\; f_1 x_1 + \ldots + f_n x_n \geq f$ for every feasible x.

Gomory Cut

Suppose Step 1 of our cutting plane algorithm gives non-integral x^*. Then there is row in Simplex tableau with

$$x_i^* + \sum_{j \notin I} y_{ij} x_j^* = y_{i0}$$

with $y_{i0} \notin \mathbb{Z}$.
Gomory Cut. Setting $f_j := y_{ij} - \lfloor y_{ij} \rfloor$, $f := y_{i0} - \lfloor y_{i0} \rfloor$, we get:

$$\sum_{j \notin I} f_j x_j \geq f. \qquad (*)$$

$(*)$ is fulfilled for all feasible x but not for x^*: $\sum_{j \notin I} f_j x_j^* = 0 < f$.

Contents of this Lecture

Example

Consider the following problem:

$$\max \ 3x_1 + 4x_2$$

subject to

$$\begin{aligned} \frac{2}{5}x_1 + x_2 &\leq 3 \\ \frac{2}{5}x_1 - \frac{2}{5}x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{Z}. \end{aligned}$$

Example

Step 1. Convert maximisation objective into minimisation.

$$\min \; x_0 = -3x_1 - 4x_2$$

subject to

$$\begin{aligned} \frac{2}{5}x_1 + x_2 &\leq 3 \\ \frac{2}{5}x_1 - \frac{2}{5}x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{Z}. \end{aligned}$$

Example

Step 1. Scale the equations of the problem.

$$\min \; x_0 = -3x_1 - 4x_2$$

subject to

$$\begin{aligned} \frac{2}{5}x_1 + x_2 &\leq 3 && (*5) \\ \frac{2}{5}x_1 - \frac{2}{5}x_2 &\leq 1 && (*5) \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{Z}. \end{aligned}$$

Example

Step 1. Scale the equations of the problem.

$$\min \ x_0 = -3x_1 - 4x_2$$

subject to

$$\begin{aligned} 2x_1 + 5x_2 &\leq 15 \\ 2x_1 - 2x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \\ x_1, x_2 &\in \mathbb{Z}. \end{aligned}$$

Example

Step 1. Insert integral slack variables.

$$\min \; x_0 = -3x_1 - 4x_2$$

subject to

$$\begin{aligned} 2x_1 + 5x_2 + x_3 \quad\quad &= 15 \\ 2x_1 - 2x_2 \quad\quad + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \\ x_1, x_2, x_3, x_4 &\in \mathbb{Z}. \end{aligned}$$

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
x_0	3	4			0
x_3	2	5	1		15
x_4	2	-2		1	5

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
x_0	3	4			0
x_3	2	5	1		15
x_4	2	-2		1	5

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
x_0	3	4			0
x_3	2	5	1		15
x_4	2	-2		1	5

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
x_0	3	4			0
x_3	2	5	1		15
x_4	2	-2		1	5
x_0	$\frac{7}{5}$	$-\frac{4}{5}$			-12
x_2	$\frac{2}{5}$	1	$\frac{1}{5}$		3
x_4	$\frac{14}{5}$		$\frac{2}{5}$	1	11

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
x_0	$\frac{7}{5}$	$-\frac{4}{5}$			-12
x_2	$\frac{2}{5}$	1	$\frac{1}{5}$		3
x_4	$\frac{14}{5}$		$\frac{2}{5}$	1	11

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
x_0	$\frac{7}{5}$	$-\frac{4}{5}$			-12
x_2	$\frac{2}{5}$	1	$\frac{1}{5}$		3
x_4	$\frac{14}{5}$		$\frac{2}{5}$	1	11

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
x_0	$\frac{7}{5}$	$-\frac{4}{5}$			-12
x_2	$\frac{2}{5}$	1	$\frac{1}{5}$		3
x_4	$\frac{14}{5}$		$\frac{2}{5}$	1	11

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	RHS
x_0	$\frac{7}{5}$	$-\frac{4}{5}$			-12
x_2	$\frac{2}{5}$	1	$\frac{1}{5}$		3
x_4	$\frac{14}{5}$		$\frac{2}{5}$	1	11
x_0			-1	$-\frac{1}{2}$	$-\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$	$\frac{55}{14}$

Solution optimal; Simplex stops.

Example

Step 3. Generate cut based on x_1 row.

$$
\begin{aligned}
& \frac{1}{7}x_3 + \frac{5}{14}x_4 \geq \frac{13}{14} \\
\Leftrightarrow \quad & \frac{1}{7}(15 - 2x_1 - 5x_2) + \frac{5}{14}(5 - 2x_1 - 2x_2) \geq \frac{13}{14} \\
\Leftrightarrow \quad & x_1 \leq 3
\end{aligned}
$$

Introduce new variable x_5 with

$$x_5 = -\frac{13}{14} + \frac{1}{7}x_3 + \frac{5}{14}x_4 = 3 - x_1.$$

Add this cut to the problem and go back to Step (1).

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_0			-1	$-\frac{1}{2}$		-$\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$		$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$		$\frac{55}{14}$
ζ			$\frac{1}{7}$	$\frac{5}{14}$	-1	$\frac{13}{14}$

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_0			-1	$-\frac{1}{2}$		$-\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$		$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$		$\frac{55}{14}$
ζ			$\frac{1}{7}$	$\frac{5}{14}$	-1	$\frac{13}{14}$

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	x_5	RHS
x_0			-1	$-\frac{1}{2}$		-$\frac{35}{2}$
x_2		1	$\frac{1}{7}$	$-\frac{1}{7}$		$\frac{10}{7}$
x_1	1		$\frac{1}{7}$	$\frac{5}{14}$		$\frac{55}{14}$
ζ			$\frac{1}{7}$	$\frac{5}{14}$	-1	$\frac{13}{14}$
x_0			$-\frac{4}{5}$		$-\frac{7}{5}$	$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$	$\frac{9}{5}$
x_1	1				1	3
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$	$\frac{13}{5}$

Solution optimal; Simplex stops.

Example

Step 3. Generate cut based on x_2 row.

$$\begin{aligned} & \frac{1}{5}x_3 + \frac{3}{5}x_5 \geq \frac{4}{5} \\ \Leftrightarrow \quad & \frac{1}{5}(15 - 2x_1 - 5x_2) + \frac{3}{5}(3 - x_1) \geq \frac{4}{5} \\ \Leftrightarrow \quad & x_1 + x_2 \leq 4 \end{aligned}$$

Introduce new variable x_6 with

$$x_6 = \frac{1}{5}x_3 + \frac{3}{5}x_5 - \frac{4}{5} = 4 - x_1 - x_2$$

Add this cut to the problem and go back to Step (1).

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0			$-\frac{4}{5}$		$-\frac{7}{5}$		$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$		$\frac{9}{5}$
x_1	1				1		3
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$		$\frac{13}{5}$
ζ			$\frac{1}{5}$		$\frac{3}{5}$	-1	$\frac{4}{5}$

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0			$-\frac{4}{5}$		$-\frac{7}{5}$		$-\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$		$\frac{9}{5}$
x_1	1				1		3
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$		$\frac{13}{5}$
ζ			$\frac{1}{5}$		$\frac{3}{5}$	-1	$\frac{4}{5}$

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_0			$\frac{4}{5}$		$\frac{7}{5}$		$\frac{81}{5}$
x_2		1	$\frac{1}{5}$		$-\frac{2}{5}$		$\frac{9}{5}$
x_1	1				1		3
x_4			$\frac{2}{5}$	1	$-\frac{14}{5}$		$\frac{13}{5}$
ζ			$\frac{1}{5}$		$\frac{3}{5}$	-1	$\frac{4}{5}$
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$	$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$	$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$	$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$	$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$	$\frac{4}{3}$

Solution optimal; Simplex stops.

Example

Step 3. Generate cut based on x_2 row.

$$\begin{aligned} & \frac{1}{3}x_3 + \frac{1}{3}x_6 \geq \frac{1}{3} \\ \Leftrightarrow \quad & \frac{1}{3}(15 - 2x_1 - 5x_2) + \frac{1}{3}(4 - x_1 - x_2) \geq \frac{1}{3} \\ \Leftrightarrow \quad & x_1 + 2x_2 \leq 6 \end{aligned}$$

Introduce new variable x_7 with

$$x_7 = \frac{1}{3}x_3 + \frac{1}{3}x_6 - \frac{1}{3} = 6 - x_1 - 2x_2$$

Add this cut to the problem and go back to Step (1).

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$		$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$		$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$		$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$		$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$		$\frac{4}{3}$
ζ			$\frac{1}{3}$			$\frac{1}{3}$	-1	$\frac{1}{3}$

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$		$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$		$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$		$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$		$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$		$\frac{4}{3}$
ζ			$\frac{1}{3}$			$\frac{1}{3}$	-1	$\frac{1}{3}$

Example

Step 1. Solve continuous relaxation of problem.

BV	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0			$-\frac{1}{3}$			$-\frac{7}{3}$		$-\frac{43}{3}$
x_2		1	$\frac{1}{3}$			$-\frac{2}{3}$		$\frac{7}{3}$
x_1	1		$-\frac{1}{3}$			$\frac{5}{3}$		$\frac{5}{3}$
x_4			$\frac{4}{3}$	1		$-\frac{14}{3}$		$\frac{19}{3}$
x_5			$\frac{1}{3}$		1	$-\frac{5}{3}$		$\frac{4}{3}$
ζ			$\frac{1}{3}$			$\frac{1}{3}$	-1	$\frac{1}{3}$
x_0						-2	-1	-14
x_2		1				-1	1	2
x_1	1					2	-1	2
x_4				1		-6	4	5
x_5					1	-2	1	1
x_3			1			1	-3	1