Chapter 12

Lecture 16 - Parseval's Identity

Lemma 12.1 *(A version of Parseval's Identity)*

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ $0 < x < L$. *Then* $\boxed{\frac{2}{L}\int_0^L \left[f(x)\right]^2 dx = \sum_{n=1}^{\infty} b_n^2.}$

Proof:

$$\int_0^L \left[f(x)\right]^2 dx = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx \quad (12.1)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \cdot \delta_{mn} \cdot \frac{L}{2} = \frac{L}{2}\sum_{n=1}^{\infty} b_n^2. \quad (12.2)$$

For a full Fourier Series on $[-L, L]$ Parseval's Theorem assumes the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \quad (12.3)$$

$$\frac{1}{L}\int_{-L}^{L} \left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2. \quad (12.4)$$

Example 12.2 Recall for $x \in [0, 2]$ $f(x) = x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right)$.

Therefore

$$\begin{aligned} \tfrac{2}{L}\int_0^L \big(f(x)\big)^2\,dx &= \tfrac{2}{2}\int_0^2 x^2\,dx = \left(\tfrac{4}{\pi}\right)^2 \sum_{n=1}^{\infty}\tfrac{1}{n^2} \\ \Rightarrow \quad \tfrac{x^3}{3}\Big|_0^2 &= \left(\tfrac{4}{\pi}\right)^2 \sum_{n=1}^{\infty}\tfrac{1}{n^2} \\ \tfrac{\pi^2}{6} &= \sum_{n=1}^{\infty}\tfrac{1}{n^2} \end{aligned} \tag{12.5}$$

Note: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n)^2} = \frac{1}{2^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{24}$.

Also note that

$$\begin{aligned} \tfrac{\pi^2}{6} = \sum_{n=1}^{\infty}\tfrac{1}{n^2} &= \overbrace{\sum_{m=1}^{\infty}\tfrac{1}{(2m)^2}}^{\text{evens}} + \overbrace{\sum_{m=0}^{\infty}\tfrac{1}{(2m+1)^2}}^{\text{odds}} \\ &= \tfrac{\pi^2}{24} + \sum_{m=0}^{\infty}\tfrac{1}{(2m+1)^2} \end{aligned}$$

Therefore

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}. \tag{12.6}$$

12.1 Geometric Interpretation of Parseval's Formula

$$\mathbf{f} = b_1\hat{\mathrm{e}}_1 + b_2\hat{\mathrm{e}}_2 \tag{12.7}$$

$$|\mathbf{f}|^2 = \mathbf{f}\cdot\mathbf{f} = b_1^2\hat{\mathrm{e}}_1\cdot\hat{\mathrm{e}}_1 + b_2^2\hat{\mathrm{e}}_2\cdot\hat{\mathrm{e}}_2 \tag{12.8}$$

$$= b_1^2 + b_2^2 \qquad \text{Pythagoras' Theorem} \tag{12.9}$$

For Fourier Sine Components:

$$\frac{2}{L}\int_0^L \big(f(x)\big)^2\,dx = \sum_{n=1}^{\infty} b_n^2. \tag{12.10}$$

Example 12.3 Consider $f(x) = x^2$ $-\pi < x < \pi$. The Fourier Series Expansion is:

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx). \tag{12.11}$$

n	1	2	3	4
$\cos\left(\frac{n\pi}{2}\right)$	0	-1	0	1

Let

$$\begin{aligned} x = \tfrac{\pi}{2} \quad \Rightarrow \quad \tfrac{\pi^2}{4} &= \tfrac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\tfrac{(-1)^n}{n^2}\cos\left(\tfrac{n\pi}{2}\right) \\ -\tfrac{\pi^2}{12} &= 4\sum_{k=1}^{\infty}\tfrac{(-1)^k}{(2k)^2} \end{aligned} \tag{12.12}$$

Therefore

$$\frac{\pi^2}{12} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}. \tag{12.13}$$

By Parseval's Formula:

$$\begin{aligned} \tfrac{2}{\pi}\int_0^{\pi} x^4\,dx &= 2\left(\tfrac{\pi^2}{3}\right)^2 + 16\sum_{n=1}^{\infty}\tfrac{1}{n^4} \qquad \tfrac{9-5}{45} = \tfrac{4}{45} = \tfrac{8}{90} \\ \tfrac{2}{\pi}\left.\tfrac{x^5}{5}\right|_0^{\pi} &= \tfrac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\tfrac{1}{n^4} \qquad \tfrac{1}{90} \end{aligned} \tag{12.14}$$

Therefore

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4} = \delta?(4). \tag{12.15}$$