Chapter 6

Numerical Integration

6.1 Introduction

After transformation to a canonical element Ω_0, typical integrals in the element stiffness or mass matrices (*cf.* (5.5.8)) have the forms

$$\mathbf{Q} = \iint_{\Omega_0} \alpha(\xi,\eta)\mathbf{N}_s\mathbf{N}_t^T \det(\mathbf{J}_e) d\xi d\eta, \tag{6.1.1a}$$

where $\alpha(\xi,\eta)$ depends on the coefficients of the partial differential equation and the transformation to Ω_0 (*cf.* Section 5.4). The subscripts s and t are either nil, ξ, or η implying no differentiation, differentiation with respect to ξ, or differentiation with respect to η, respectively. Assuming that $\mathbf{N}$ has the form

$$\mathbf{N}^T = [N_1, N_2, \ldots, N_{n_p}], \tag{6.1.1b}$$

then (6.1.1a) may be written in the more explicit form

$$\mathbf{Q} = \iint_{\Omega_0} \alpha(\xi,\eta) \begin{bmatrix} (N_1)_s(N_1)_t & (N_1)_s(N_2)_t & & (N_1)_s(N_{n_p})_t \\ (N_2)_s(N_1)_t & (N_2)_s(N_2)_t & & (N_2)_s(N_{n_p})_t \\ & & \ddots & \\ (N_{n_p})_s(N_1)_t & (N_{n_p})_s(N_2)_t & & (N_{n_p})_s(N_{n_p})_t \end{bmatrix} \det(\mathbf{J}_e) d\xi d\eta. \tag{6.1.1c}$$

Integrals of the form (6.1.1b) may be evaluated exactly when the coordinate transformation is linear ($\mathbf{J}_e$ is constant) and the coefficients of the differential equation are constant (*cf.* Problem 1 at the end of this section). With certain coefficient functions and transformations it may be possible to evaluate (6.1.1b) exactly by symbolic integration; however, we'll concentrate on numerical integration because:

- it can provide exact results in simple situations (*e.g.*, when α and $\mathbf{J}_e$ are constants) and

- exact integration is not needed to achieve the optimal convergence rate of finite element solutions ([2, 9, 11], and Chapter 7).

Integration is often called *quadrature* in one dimension and *cubature* in higher dimensions; however, we'll refer to all numerical approximations as *quadrature rules*. We'll consider integrals and quadrature rules of the form

$$I = \iint\limits_{\Omega_0} f(\xi,\eta)d\xi d\eta \approx \sum_{i=1}^{n} W_i f(\xi_i,\eta_i). \tag{6.1.2a}$$

where W_i, are the quadrature rule's *weights* and (ξ_i, η_i) are the *evaluation points*, $i = 1, 2, \ldots, n$. Of course, we'll want to appraise the accuracy of the approximate integration and this is typically done by indicating those polynomials that are integrated exactly.

Definition 6.1.1. The integration rule (6.1.2a) is *exact to order* q if it is exact when $f(\xi,\eta)$ is any polynomial of degree q or less.

When the integration rule is exact to order q and $f(\xi,\eta) \in H^{q+1}(\Omega_0)$, the error

$$E = I - \sum_{i=1}^{n} W_i f(\xi_i,\eta_i) \tag{6.1.2b}$$

satisfies an estimate of the form

$$E \leq C||f(\cdot,\cdot)||_{q+1}. \tag{6.1.2c}$$

Example 6.1.1. Applying (6.1.2) to (6.1.1a) yields

$$\mathbf{Q} \approx \sum_{i=1}^{n} W_i \alpha(\xi_i,\eta_i)\mathbf{N}(\xi_i,\eta_i)\mathbf{N}^T(\xi_i,\eta_i)\det(\mathbf{J}_e(\xi_i,\eta_i)).$$

Thus, the integrand at the evaluation points is summed relative to the weights to approximate the given integral.

Problems

1. A typical term of an element stiffness or mass matrix has the form

$$\iint\limits_{\Omega_0} \xi^i \eta^j d\xi d\eta, \qquad i,j, \geq 0.$$

Evaluate this integral when Ω_0 is the canonical square $[-1,1]\times[-1,1]$ and the canonical right $45°$ unit triangle.

6.2 One-Dimensional Gaussian Quadrature

Although we are primarily interested in two- and three-dimensional quadrature rules, we'll set the stage by studying one-dimensional integration. Thus, consider the one-dimensional equivalent of (6.1.2) on the canonical $[-1, 1]$ element

$$I = \int_{-1}^{1} f(\xi) d\xi = \sum_{i=1}^{n} W_i f(\xi_i) + E. \tag{6.2.1}$$

Most classical quadrature rules have this form. For example,the trapezoidal rule

$$I \approx f(-1) + f(1)$$

has the form (6.2.1) with $n = 2$, $W_1 = W_2 = 1$, $-\xi_1 = \xi_2 = 1$, and

$$E = -\frac{2f''(\sigma)}{3}, \qquad \sigma \in (-1, 1).$$

Similarly, Simpson's rule

$$I \approx \frac{1}{3}[f(-1) + 4f(0) + f(1)]$$

has the form (6.2.1) with $n = 3$, $W_1 = W_2/4 = W_3 = 1/3$, $-\xi_1 = \xi_3 = 1$, $\xi_2 = 0$, and

$$E = -\frac{f^{(iv)}(\sigma)}{90}, \qquad \sigma \in (-1, 1).$$

Gaussian quadrature is preferred to these Newton-Cotes formulas for finite element applications because they have fewer function evaluations for a given order. With Gaussian quadrature, the weights and evaluation points are determined so that the integration rule is exact ($E = 0$) to as high an order as possible. Since there are $2n$ unknown weights and evaluation points, we expect to be able to make (6.2.1) exact to order $2n - 1$. This problem has been solved [3, 6] and the evaluation points ξ_i, $i = 1, 2, \ldots, n$, are the roots of the Legendre polynomial of degree n (*cf.* Section 2.5). The weights W_i, $i = 1, 2, \ldots, n$, called *Christoffel weights*, are also known and are tabulated with the evaluation points in Table 6.2.1 for n ranging from 1 to 6. A more complete set of values appear in Abromowitz and Stegun [1].

Example 6.2.1. The derivation of the two-point ($n = 2$) Gauss quadrature rule is given as Problem 1 at the end of this section. From Table 6.2.1 we see that $W_1 = W_2 = 1$ and $-\xi_1 = \xi_2 = 1/\sqrt{3}$. Thus, the quadrature rule is

$$\int_{-1}^{1} f(\xi) d\xi \approx f(-1/\sqrt{3}) + f(1/\sqrt{3}).$$

This formula is exact to order three; thus the error is proportional to the fourth derivative of f (*cf.* Theorem 6.2.1, Example 6.2.4, and Problem 2 at the end of this section).

n	$\pm\xi_i$	W_i
1	0.00000 00000 00000	2.00000 00000 00000
2	0.57735 02691 89626	1.00000 00000 00000
3	0.00000 00000 00000	0.88888 88888 88889
	0.77459 66692 41483	0.55555 55555 55556
4	0.33998 10435 84856	0.65214 51548 62546
	0.86113 63115 94053	0.34785 48451 37454
5	0.00000 00000 00000	0.56888 88888 88889
	0.53846 93101 05683	0.47862 86704 99366
	0.90617 98459 38664	0.23692 68850 56189
6	0.23861 91860 83197	0.46791 39345 72691
	0.66120 93864 66265	0.36076 15730 48139
	0.93246 95142 03152	0.17132 44923 79170

Table 6.2.1: Christoffel weights W_i and roots ξ_i, $i = 1, 2, \ldots, n$, for Legendre polynomials of degrees 1 to 6 [1].

Example 6.2.2. Consider evaluating the integral

$$I = \int_0^1 e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \text{erf}(1) = 0.74682413281243 \qquad (6.2.2)$$

by Gauss quadrature. Let us transform the integral to $[-1, 1]$ using the mapping

$$\xi = 2x - 1$$

to get

$$I = \frac{1}{2} \int_{-1}^1 e^{-(\frac{1+\xi}{2})^2} d\xi.$$

The two-point Gaussian approximation is

$$I \approx \tilde{I} = \frac{1}{2}[e^{-(\frac{1-1/\sqrt{3}}{2})^2} + e^{-(\frac{1+1/\sqrt{3}}{2})^2}].$$

Other approximations follow in similar order.

Errors $I - \tilde{I}$ when I is approximated by Gaussian quadrature to obtain $\tilde{I}$ appear in Table 6.2.2 for n ranging from 1 to 6. Results using the trapezoidal and Simpson's rules are also presented. The two- and three-point Gaussian rules have higher orders than the corresponding Newton-Cotes formulas and this leads to smaller errors for this example.

n	Gauss Rules Error	Newton Rules Error
1	3.198(- 2)	
2	-2.294(- 4)	-6.288(- 2)
3	-9.549(- 6)	3.563(- 4)
4	3.353(- 7)	
5	-6.046(- 9)	
6	7.772(-11)	

Table 6.2.2: Errors in approximating the integral of Example 6.2.2 by Gauss quadrature, the trapezoidal rule ($n = 2$, right) and Simpson's rule ($n = 3$, right). Numbers in parentheses indicate a power of ten.

Example 6.2.3. Composite integration formulas, where the domain of integration $[a, b]$ is divided into N subintervals of width

$$\Delta x_j = x_j - x_{j-1}, \qquad j = 1, 2, \ldots, N,$$

are not needed in finite element applications, except, perhaps, for postprocessing. However, let us do an example to illustrate the convergence of a Gaussian quadrature formula. Thus, consider

$$I = \int_a^b f(x)dx = \sum_{j=1}^{n} I_j$$

where

$$I_j = \int_{x_{j-1}}^{x_j} f(x)dx.$$

The linear mapping

$$x = x_{j-1}\frac{1-\xi}{2} + x_j\frac{1+\xi}{2}$$

transforms $[x_{j-1}, x_j]$ to $[-1, 1]$ and

$$I_j = \frac{\Delta x_j}{2}\int_{-1}^{1} f(x_{j-1}\frac{1-\xi}{2} + x_j\frac{1+\xi}{2})d\xi.$$

Approximating I_j by Gauss quadrature gives

$$I_j \approx \frac{\Delta x_j}{2}\sum_{i=1}^{n} W_i f(x_{j-1}\frac{1-\xi_i}{2} + x_j\frac{1+\xi_i}{2}).$$

We'll approximate (6.2.2) using composite two-point Gauss quadrature; thus,

$$I_j = \frac{\Delta x_j}{2}[e^{-(x_{j-1/2}-\Delta x_j/(2\sqrt{3}))^2} + e^{-(x_{j-1/2}+\Delta x_j/(2\sqrt{3}))^2}],$$

where $x_{j-1/2} = (x_j + x_{j-1})/2$. Assuming a uniform partition with $\Delta x_j = 1/N$, $j = 1, 2, \ldots, N$, the composite two-point Gauss rule becomes

$$I \approx \frac{1}{2N} \sum_{j=1}^{n} [e^{-(x_{j-1/2} - 1/(2N\sqrt{3}))^2} + e^{-(x_{j-1/2} + 1/(2N\sqrt{3}))^2}].$$

The composite Simpson's rule,

$$I \approx \frac{1}{3N} [1 + 4 \sum_{i=1,3}^{N-1} e^{-x_j} + 2 \sum_{i=2,4}^{N-2} e^{-x_j} + e^{-1}]$$

on $N/2$ subintervals of width $2\Delta x$ has an advantage relative to the composite Gauss rule since the function evaluations at the even-indexed points combine.

The number of function evaluations and errors when (6.2.2) is solved by the composite two-point Gauss and Simpson's rules are recorded in Table 6.2.3. We can see that both quadrature rules are converging as $O(1/N^4)$ ([6], Chapter 7). The computations were done in single precision arithmetic as opposed to those appearing in Table 6.2.2, which were done in double precision. With single precision, round-off error dominates the computation as N increases beyond 16 and further reductions of the error are impossible. With function evaluations defined as the number of times that the exponential is evaluated, errors for the same number of function evaluations are comparable for Gauss and Simpson's rule quadrature. As noted earlier, this is due to the combination of function evaluations at the ends of even subintervals. Discontinuous solution derivatives at inter-element boundaries would prevent such a combination with finite element applications.

N	Gauss Rules		Simpson's Rule	
	Fn. Eval.	Abs. Error	Fn. Eval.	Abs. Error
2	4	0.208(- 4)	3	0.356(- 3)
4	8	0.161(- 5)	5	0.309(- 4)
8	16	0.358(- 6)	9	0.137(- 5)
16	32	0.364(- 5)	17	0.244(-5)

Table 6.2.3: Comparison of composite two-point Gauss and Simpson's rule approximations for Example 6.2.3. The absolute error is the magnitude of the difference between the exact and computational result. The number of times that the exponential function is evaluated is used as a measure of computational effort.

As we may guess, estimates of errors for Gauss quadrature use the properties of Legendre polynomials (*cf.* Section 2.5). Here is a typical result.

Theorem 6.2.1. *Let $f(\xi) \in C^{2n}[-1,1]$, then the quadrature rule (6.2.1) is exact to order $2n-1$ if ξ_i, $i = 1, 2, \ldots, n$, are the roots of $P_n(\xi)$, the nth-degree Legendre polynomial, and the corresponding Christoffel weights satisfy*

$$W_i = \frac{1}{P_n'(\xi_i)} \int_{-1}^{1} \frac{P_n(\xi)}{\xi - \xi_i} d\xi, \qquad i = 1, 2, \ldots, n. \tag{6.2.3a}$$

Additionally, there exists a point $\zeta \in (-1,1)$ such that

$$E = \frac{f^{(2n)}(\zeta)}{2n!} \int_{-1}^{1} \prod_{i=1}^{n} (\xi - \xi_i)^2 d\xi. \tag{6.2.3b}$$

Proof. cf. [6], Sections 7.3, 4. □

Example 6.2.4. Using the entries in Table 6.2.1 and (6.2.3b), the discretization error of the two-point ($n = 2$) Gauss quadrature rule is

$$E = \frac{f^{iv}(\zeta)}{4!} \int_{-1}^{1} (\xi + \frac{1}{\sqrt{3}})^2 (\xi - \frac{1}{\sqrt{3}})^2 d\xi = \frac{f^{iv}(\zeta)}{135}, \qquad \zeta \in (-1,1).$$

Problems

1. Calculate the weights W_1 and W_2 and the evaluation points ξ_1 and ξ_2 so that the two-point Gauss quadrature rule

$$\int_{-1}^{1} f(x) \approx W_1 f(\xi_1) + W_2 f(\xi_2)$$

   is exact to as high an order as possible. This should be done by a direct calculation without using the properties of Legendre polynomials.

2. Lacking the precise information of Theorem 6.2.1, we may infer that the error in the two-point Gauss quadrature rule is proportional to the fourth derivative of $f(\xi)$ since cubic polynomials are integrated exactly. Thus,

$$E = Cf^{iv}(\zeta), \qquad \zeta \in (-1,1).$$

   We can determine the error coefficient C by evaluating the formula for any function $f(x)$ whose fourth derivative does not depend on the location of the unknown point ζ. In particular, any quartic polynomial has a constant fourth derivative; hence, the value of ζ is irrelevant. Select an appropriate quartic polynomial and show that $C = 1/135$ as in Example 6.2.4.

6.3 Multi-Dimensional Quadrature

Integration on square elements usually relies on tensor products of the one-dimensional formulas illustrated in Section 6.2. Thus, the application of (6.2.1) to a two-dimensional integral on a canonical $[-1,1] \times [-1,1]$ square element yields the approximation

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta \approx \int_{-1}^{1}\sum_{i=1}^{n} W_i f(\xi_i,\eta)d\eta = \sum_{i=1}^{n} W_i \int_{-1}^{1} f(\xi_i,\eta)d\eta$$

and

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta \approx \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j f(\xi_i,\eta_j). \tag{6.3.1}$$

Error estimates follow the one-dimensional analysis.

Tensor-product formulas are not optimal in the sense of using the fewest function evaluations for a given order. Exact integration of a quintic polynomial by (6.3.1) would require $n = 3$ or a total of 9 points. A complete quintic polynomial in two dimensions has 21 monomial terms; thus, a direct (non-tensor-product) formula of the form

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta \approx \sum_{i=1}^{n} W_i f(\xi_i,\eta_i)$$

could be made exact with only 7 points. The 21 coefficients W_i, ξ_i, η_i, $i = 1, 2, \ldots, 7$, could potentially be determined to exactly integrate all of the monomial terms.

Non-tensor-product formulas are complicated to derive and are not known to very high orders. Orthogonal polynomials, as described in Section 6.2, are unknown in two and three dimensions. Quadrature rules are generally derived by a method of undetermined coefficients. We'll illustrate this approach by considering an integral on a canonical right 45° triangle

$$I = \iint_{\Omega_0} f(\xi,\eta)d\xi d\eta = \sum_{i=1}^{n} W_i f(\xi_i,\eta_i) + E. \tag{6.3.2}$$

Example 6.3.1. Consider the one-point quadrature rule

$$\iint_{\Omega_0} f(\xi,\eta)d\xi d\eta = W_1 f(\xi_1,\eta_1) + E. \tag{6.3.3}$$

Since there are three unknowns W_1, ξ_1, and η_1, we expect (6.3.3) to be exact for any linear polynomial. Integration is a linear operator; hence, it suffices to ensure that (6.3.3) is exact for the monomials 1, ξ, and η. Thus,

- If $f(\xi,\eta) = 1$:

$$\int_0^1 \int_0^{1-\xi} (1) d\eta d\xi = \frac{1}{2} = W_1.$$

- If $f(\xi,\eta) = \xi$:

$$\int_0^1 \int_0^{1-\xi} (\xi) d\eta d\xi = \frac{1}{6} = W_1 \xi_1.$$

- If $f(\xi,\eta) = \eta$:

$$\int_0^1 \int_0^{1-\xi} (\eta) d\eta d\xi = \frac{1}{6} = W_1 \eta_1.$$

The solution of this system is $W_1 = 1/2$ and $\xi_1 = \eta_1 = 1/3$; thus, the one-point quadrature rule is

$$\iint_{\Omega_0} f(\xi,\eta) d\xi d\eta = \frac{1}{2} f(\frac{1}{3},\frac{1}{3}) + E. \tag{6.3.4}$$

As expected, the optimal evaluation point is the centroid of the triangle.

A bound on the error E may be obtained by expanding $f(\xi,\eta)$ in a Taylor's series about some convenient point $(\xi_0,\eta_0) \in \Omega_0$ to obtain

$$f(\xi,\eta) = p_1(\xi,\eta) + R_1(\xi,\eta) \tag{6.3.5a}$$

where

$$p_1(\xi,\eta) = f(\xi_0,\eta_0) + [(\xi-\xi_0)\frac{\partial}{\partial\xi} + (\eta-\eta_0)\frac{\partial}{\partial\eta}] f(\xi_0,\eta_0) \tag{6.3.5b}$$

and

$$R_1(\xi,\eta) = \frac{1}{2}[(\xi-\xi_0)\frac{\partial}{\partial\xi} + (\eta-\eta_0)\frac{\partial}{\partial\eta}]^2 f(\theta,\omega), \qquad (\theta,\omega) \in \Omega_0. \tag{6.3.5c}$$

Integrating (6.3.5a) using (6.3.4)

$$E = \iint_{\Omega_0} [p_1(\xi,\eta) + R_1(\xi,\eta)] d\xi d\eta - \frac{1}{2}[p_1(\frac{1}{3},\frac{1}{3}) + R_1(\frac{1}{3},\frac{1}{3})].$$

Since (6.3.4) is exact for linear polynomials

$$E = \iint_{\Omega_0} R_1(\xi,\eta) d\xi d\eta - \frac{1}{2} R_1(\frac{1}{3},\frac{1}{3}).$$

Not being too precise, we take an absolute value of the above expression to obtain

$$|E| \le \iint_{\Omega_0} |R_1(\xi,\eta)| d\xi d\eta + \frac{1}{2}|R_1(\frac{1}{3},\frac{1}{3})|.$$

For the canonical element, $|\xi - \xi_0| \leq 1$ and $|\eta - \eta_0| \leq 1$; hence,

$$|R_1(\xi, \eta)| \leq 2 \max_{|\boldsymbol{\kappa}|=2} ||D^{\boldsymbol{\kappa}} f||_{\infty,0}$$

where

$$||f||_{\infty,0} = \max_{(\xi,\eta)\in\Omega_0} |f(\xi,\eta)|.$$

Since the area of Ω_0 is 1/2,

$$|E| \leq 2 \max_{|\boldsymbol{\kappa}|=2} ||D^{\boldsymbol{\kappa}} f||_{\infty,0}. \tag{6.3.6}$$

Errors for other quadrature formulas follow the same derivation ([6], Section 7.7).

Two-dimensional integrals on triangles are conveniently expressed in terms of triangular coordinates as

$$\iint_{\Omega_e} f(x,y)dxdy = A_e \sum_{i=1}^{n} W_i f(\zeta_1^i, \zeta_2^i, \zeta_3^i) + E \tag{6.3.7}$$

where $(\zeta_1^i, \zeta_2^i, \zeta_3^i)$ are the triangular coordinates of evaluation point i and A_e is the area of triangle e. Symmetric quadrature formulas for triangles have appeared in several places. Hammer *et al.* [5] developed formulas on triangles, tetrahedra, and cones. Dunavant [4] presents formulas on triangles which are exact to order 20; however, some formulas have evaluation points that are outside of the triangle. Sylvester [10] developed tensor-product formulas for triangles. We have listed some quadrature rules in Table 6.3.1 that also appear in Dunavant [4], Strang and Fix [9], and Zienkiewicz [12]. A multiplication factor M indicates the number of permutations associated with an evaluation point having a weight W_i. The factor $M = 1$ is associated with an evaluation point at the triangle's centroid $(1/3, 1/3, 1/3)$, $M = 3$ indicates a point on a median line, and $M = 6$ indicates an arbitrary point in the interior. The factor p indicates the order of the quadrature rule; thus, $E = O(h^{p+1})$ where h is the maximum edge length of the triangle.

Example 6.3.2. Using the data in Table 6.3.1 with (6.3.7), the three-point quadrature rule on the canonical triangle is

$$\iint_{\Omega_0} f(\xi,\eta)d\xi d\eta = \frac{1}{6}[f(2/3,1/6,1/6) + f(1/6,1/6,2/3) + f(1/6,2/3,1/6)] + E.$$

The multiplicative factor of 1/6 arises because the area of the canonical element is 1/2 and all of the weights are 1/3. The quadrature rule can be written in terms of the canonical variables by setting $\zeta_2 = \xi$ and $\zeta_3 = \eta$ (*cf.* (4.2.6) and (4.2.7)). The discretization error associated with this quadrature rule is $O(h^3)$.

n	W_i	ζ_1^i	ζ_2^i, ζ_3^i	M	p
1	1.000000000000000	0.333333333333333	0.333333333333333		1
			0.333333333333333	1	
3	0.333333333333333	0.666666666666667	0.166666666666667		2
			0.166666666666667	3	
4	-0.562500000000000	0.333333333333333	0.333333333333333		3
			0.333333333333333	1	
	0.520833333333333	0.600000000000000	0.200000000000000		
			0.200000000000000	3	
6	0.109951743655322	0.816847572980459	0.091576213509771		4
			0.091576213509771	3	
	0.223381589678011	0.108103018168070	0.445948490915965		
			0.445948490915965	3	
7	0.225000000000000	0.333333333333333	0.333333333333333		5
			0.333333333333333	1	
	0.125939180544827	0.797426985353087	0.101286507323456		
			0.101286507323456	3	
	0.132394152788506	0.059715871789770	0.470142064105115		
			0.470142064105115	3	
12	0.050844906370207	0.873821971016996	0.063089014491502		6
			0.063089014491502	3	
	0.116786275726379	0.501426509658179	0.249286745170910		
			0.249286745170910	3	
	0.082851075618374	0.636502499121399	0.310352451033785		
			0.053145049844816	6	
13	-0.149570044467670	0.333333333333333	0.333333333333333		7
			0.333333333333333	1	
	0.175615257433204	0.479308067841923	0.260345966079038		
			0.260345966079038	3	
	0.053347235608839	0.869739794195568	0.065130102902216		
			0.065130102902216	3	
	0.077113760890257	0.638444188569809	0.312865496004875		
			0.048690315425316	6	

Table 6.3.1: Weights and evaluation points for integration on triangles [4].

Quadrature rules on tetrahedra have the form

$$\iiint_{\Omega_e} f(x,y,z)dxdydz = V_e \sum_{i=1}^{n} W_i f(\zeta_1^i, \zeta_2^i, \zeta_3^i, \zeta_4^i) + E \tag{6.3.8}$$

where V_e is the volume of Element e and $(\zeta_1^i, \zeta_2^i, \zeta_3^i, \zeta_4^i)$ are the tetrahedral coordinates of evaluation point i. Quadrature rules are presented by Jinyun [7] for methods to order six and by Keast [8] for methods to order eight. Multiplicative factors are such that $M = 1$ for an evaluation point at the centroid $(1/4, 1/4, 1/4, 1/4)$, $M = 4$ for points on the median line through the centroid and one vertex, $M = 6$ for points on a line between opposite midsides, $M = 12$ for points in the plane containing an edge an and opposite midside, and $M = 24$ for points in the interior (Figure 6.3.1).

n	W_i	ζ_1^i, ζ_2	ζ_3, ζ_4	M	p
1	1.000000000000000	0.250000000000000	0.250000000000000		1
		0.250000000000000	0.250000000000000	1	
4	0.250000000000000	0.585410196624969	0.138196601125011		2
		0.138196601125011	0.138196601125011	4	
5	-0.800000000000000	0.250000000000000	0.250000000000000		3
		0.250000000000000	0.250000000000000	1	
	0.450000000000000	0.500000000000000	0.166666666666667		
		0.166666666666667	0.166666666666667	4	
11	-0.013155555555556	0.250000000000000	0.250000000000000		4
		0.250000000000000	0.250000000000000	1	
	0.007622222222222	0.785714285714286	0.071428571428571		
		0.071428571428571	0.071428571428571	4	
	0.024888888888889	0.399403576166799	0.399403576166799		
		0.100596423833201	0.100596423833201	6	
15	0.030283678097089	0.250000000000000	0.250000000000000		5
		0.250000000000000	0.250000000000000	1	
	0.006026785714286	0.000000000000000	0.333333333333333		
		0.333333333333333	0.333333333333333	4	
	0.011645249086029	0.727272727272727	0.090909090909091		
		0.090909090909091	0.090909090909091	4	
	0.010949141561386	0.066550153573664	0.066550153573664		
		0.433449846426336	0.433449846426336	6	

Table 6.3.2: Weights and evaluation points for integration on tetrahedra [7, 8].

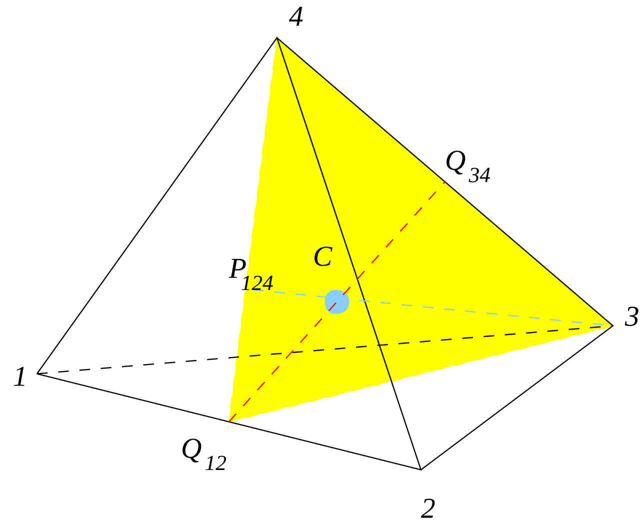

Figure 6.3.1: Some symmetries associated with the tetrahedral quadrature rules of Table 6.3.2. An evaluation point with $M = 1$ is at the centroid (C), one with $M = 4$ is on a line through a vertex and the centroid (*e.g.*, line $3 - P_{134}$), one with $M = 6$ is on a line between two midsides (*e.g.*, line $Q_{12} - Q_{34}$), and one with $M = 12$ is in a plane through two vertices and an opposite midside (*e.g.*, plane $3 - 4 - Q_{12}$)

Problems

1. Derive a three-point Gauss quadrature rule on the canonical right 45° triangle that is accurate to order two. In order to simplify the derivation, use symmetry arguments to conclude that the three points have the same weight and that they are symmetrically disposed on the medians of the triangle. Show that there are two possible formulas: the one given in Table 6.3.1 and another one. Find both formulas.

2. Show that the mapping

$$\xi = \frac{1+u}{2}, \qquad \eta = \frac{(1-u)(1+v)}{4}$$

transforms the integral (6.3.2) from the triangle Ω_0 to one on the square $-1 \leq u, v \leq 1$. Find the resulting integral and show how to approximate it using a tensor-product formula.

Bibliography

[1] M. Abromowitz and I.A. Stegun. *Handbook of Mathematical Functions*, volume 55 of *Applied Mathematics Series*. National Bureau of Standards, Gathersburg, 1964.

[2] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, New York, 1994.

[3] R.L. Burden and J.D. Faires. *Numerical Analysis*. PWS-Kent, Boston, fifth edition, 1993.

[4] D.A. Dunavant. High degree efficient symmetrical Gaussian quadrature rules for the triangle. *International Journal of Numerical Methods in Engineering*, 21:1129–1148, 1985.

[5] P.C. Hammer, O.P. Marlowe, and A.H. Stroud. Numerical integration over simplexes and cones. *Mathematical Tables and other Aids to Computation*, 10:130–137, 1956.

[6] E. Isaacson and H.B. Keller. *Analysis of Numerical Methods*. John Wiley and Sons, New York, 1966.

[7] Y. Jinyun. Symmetric Gaussian quadrature formulae for tetrahedronal regions. *Computer Methods in Applied Mechanics and Engineering*, 43:349–353, 1984.

[8] P. Keast. Moderate-degree tetrahedral quadrature formulas. *Computer Methods in Applied Mechanics and Engineering*, 55:339–348, 1986.

[9] G. Strang and G. Fix. *Analysis of the Finite Element Method*. Prentice-Hall, Englewood Cliffs, 1973.

[10] P. Sylvester. Symmetric quadrature formulae for simplexes. *Mathematics of Computation*, 24:95–100, 1970.

[11] R. Wait and A.R. Mitchell. *The Finite Element Analysis and Applications*. John Wiley and Sons, Chichester, 1985.

[12] O.C. Zienkiewicz. *The Finite Element Method.* McGraw-Hill, New York, third edition, 1977.