

MATHEMATICAL METHODS

PARTIAL DIFFERENTIAL EQUATIONS

I YEAR B.Tech

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By

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SYLLABUS OF MATHEMATICAL METHODS (as per JNTU Hyderabad)

Name of the Unit	Name of the Topic
Unit-I Solution of Linear systems	Matrices and Linear system of equations: Elementary row transformations – Rank – Echelon form, Normal form – Solution of Linear Systems – Direct Methods – LU Decomposition from Gauss Elimination – Solution of Tridiagonal systems – Solution of Linear Systems.
Unit-II Eigen values and Eigen vectors	Eigen values, Eigen vectors – properties – Condition number of Matrix, Cayley – Hamilton Theorem (without proof) – Inverse and powers of a matrix by Cayley – Hamilton theorem – Diagonalization of matrix – Calculation of powers of matrix – Model and spectral matrices.
Unit-III Linear Transformations	Real Matrices, Symmetric, skew symmetric, Orthogonal, Linear Transformation - Orthogonal Transformation. Complex Matrices, Hermitian and skew Hermitian matrices, Unitary Matrices - Eigen values and Eigen vectors of complex matrices and their properties. Quadratic forms - Reduction of quadratic form to canonical form, Rank, Positive, negative and semi definite, Index, signature, Sylvester law, Singular value decomposition.
Unit-IV Solution of Non-linear Systems	Solution of Algebraic and Transcendental Equations- Introduction: The Bisection Method – The Method of False Position – The Iteration Method - Newton –Raphson Method Interpolation: Introduction-Errors in Polynomial Interpolation - Finite differences- Forward difference, Backward differences, Central differences, Symbolic relations and separation of symbols-Difference equations – Differences of a polynomial - Newton’s Formulae for interpolation - Central difference interpolation formulae - Gauss Central Difference Formulae - Lagrange’s Interpolation formulae- B. Spline interpolation, Cubic spline.
Unit-V Curve fitting & Numerical Integration	Curve Fitting: Fitting a straight line - Second degree curve - Exponential curve - Power curve by method of least squares. Numerical Integration: Numerical Differentiation-Simpson’s 3/8 Rule, Gaussian Integration, Evaluation of Principal value integrals, Generalized Quadrature.
Unit-VI Numerical solution of ODE	Solution by Taylor’s series - Picard’s Method of successive approximation- Euler’s Method -Runge kutta Methods, Predictor Corrector Methods, Adams- Bashforth Method.
Unit-VII Fourier Series	Determination of Fourier coefficients - Fourier series-even and odd functions - Fourier series in an arbitrary interval - Even and odd periodic continuation - Half-range Fourier sine and cosine expansions.
Unit-VIII Partial Differential Equations	Introduction and formation of PDE by elimination of arbitrary constants and arbitrary functions - Solutions of first order linear equation - Non linear equations - Method of separation of variables for second order equations - Two dimensional wave equation.

CONTENTS

UNIT-VIII

PARTIAL DIFFERENTIAL EQUATIONS

- **Introduction to PDE**
- **Formation of PDE by elimination of arbitrary constants**
- **Formation of PDE by elimination of arbitrary Functions**
- **Solutions of First order Linear equations**
- **Non Linear equations (Types)**
- **Method of Separation of Variables**

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PARTIAL DIFFERENTIAL EQUATIONS

The Partial Differential Equation (PDE) corresponding to a physical system can be formed, either by eliminating the arbitrary constants or by eliminating the arbitrary functions from the given relation.

The Physical system contains arbitrary constants or arbitrary functions or both.

Equations which contain one or more partial derivatives are called Partial Differential Equations. Therefore, there must be at least two independent variables and one dependent variable.

Let us take x, y to be two independent variables and z to be dependent variable.

Order: The Order of a partial differential equation is the order of the highest partial derivative in the equation.

Degree: The degree of the highest partial derivative in the equation is the Degree of the PDE

Notations

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

Formation of Partial Differential Equation

- ▶ Formation of PDE by elimination of Arbitrary Constants
- ▶ Formation of PDE by elimination of Arbitrary Functions

Solution of a Partial Differential Equation

Let us consider a Partial Differential Equation of the form $F(x, y, z, p, q) = 0 \longrightarrow \textcircled{1}$

If it is Linear in p and q , it is called a **Linear Partial Differential Equation**

(i.e. Order and Degree of p and q is one)

If it is Not Linear in p and q , it is called as **nonlinear Partial Differential Equation**

(i.e. Order and Degree of p and q is other than one)

Consider a relation of the type $F(x, y, z, a, b) = 0$

By eliminating the arbitrary constants a and b from this equation, we get $F(x, y, z, p, q) = 0$, which is called a **complete integral** or **complete solution** of the PDE.

A solution of $F(x, y, z, p, q) = 0$ obtained by giving particular values to a and b in the complete Integral is called a **particular Integral**.

LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A Differential Equation which involves partial derivatives p and q only and no higher order derivatives is called a first order equation. If p and q have the degree one, it is called a linear partial differential equation of first order; otherwise it is called a non-linear partial equation of first order.

Ex: 1) $px + qy^2 = z$ is a linear Partial Differential Equation.

2) $p^2 + q^2 = 1$ is a non-linear Partial Differential Equation.

LAGRANGE'S LINEAR EQUATION

A linear Partial Differential Equation of order one, involving a dependent variable z and two independent variables x and y , and is of the form $Pp + Qq = R$, where P, Q, R are functions of x, y, z is called Lagrange's Linear Equation.

Solution of the Linear Equation

Consider $Pp + Qq = R$

$$\text{Now, } \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Case 1: If it is possible to separate variables then, consider any two equations, solve them by integrating. Let the solutions of these equations are $u = a, u = b$

$\therefore \varphi(u, v) = 0$ is the required solution of given equation.

Case 2: If it is not possible to separate variables then

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$$

To solve above type of problems we have following methods

► **Method of grouping:** In some problems, it is possible to solve any two of the equations

$$\frac{dx}{P} = \frac{dy}{Q} \quad (\text{or}) \quad \frac{dy}{Q} = \frac{dz}{R} \quad (\text{or}) \quad \frac{dx}{P} = \frac{dz}{R}$$

In such cases, solve the differential equation, get the solution and then substitute in the other differential equation

► **Method of Multiplier:** consider $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{0}$

In this, we have to choose l, m, n so that denominator=0. That will give us solution by integrating $l dx + m dy + n dz$.

NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A partial differential equation which involves first order partial derivatives p and q with degree higher than one and the products of p and q is called a non-linear partial differential equation.

There are six types of non-linear partial differential equations of first order as given below.

- ▶ Type I: $f(p, q) = 0$
- ▶ Type II: $f(p, q, z) = 0$
- ▶ Type III: $f_1(p, x) = f_2(y, q)$ (variable separable method)
- ▶ Type IV: Clairaut's Form
- ▶ Equation reducible to standard forms $f(x^m p, y^n q) = 0$ and $f(x^m p, y^n q, z) = 0$ and $f(pz^m, qz^m) = 0$ and $f_1(x, pz^m) = f_2(y, qz^m)$
- ▶ CHARPIT'S METHOD

Let us see in detail about these types.

Type I: $f(p, q) = 0$

Equations of the type $f(p, q) = 0$ i.e. equations containing p and q only

Let the required solution be $z = ax + by + c$

$$\therefore \frac{\partial z}{\partial x} = a \text{ and } \frac{\partial z}{\partial y} = b$$

Substituting these values in $f(p, q) = 0$, we get $f(a, b) = 0$

From this, we can obtain b in terms of a (or) a in terms of b

Let $b = \varphi(a)$, then the required solution is $z = ax + \varphi(a)y + c$

Note: Since, the given equation contains two first order partial derivatives (p & q), the final solution should contain only two constants.

Type II: $f(p, q, z) = 0$

Let us consider the **Equations of the type $f(p, q, z) = 0$** \longrightarrow ①

Let z is a function of u and $u = x + ay$

i.e. $z = z(u)$ and $u = x + ay$

$$\text{Now, } p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \cdot 1 = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du} \cdot a = a \frac{dz}{du}$$

① $\Rightarrow f\left(\frac{dz}{du}, a \frac{dz}{du}, z\right) = 0$ is the 1st order differential equation in terms of dependent variable z and independent variable u .

Solve this differential equation and finally substitute $u = x + ay$ gives the required solution.

Type III: $f_1(p, x) = f_2(y, q)$ (variable separable method)

Let us consider the differential equation is of the form $f_1(p, x) = f_2(y, q)$

Let $f_1(p, x) = f_2(y, q) = a$ (say)

Now $f_1(p, x) = a \Rightarrow p = \Psi_1(x)$ (I.e. writing p in terms of x)

$f_2(y, q) = a \Rightarrow q = \Psi_2(y)$ (I.e. writing q in terms of y)

Now, $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$$= p dx + q dy$$

$$\Rightarrow dz = \Psi_1(x)dx + \Psi_2(y)dy$$

By Integrating this, we get the required solution.

Note: This method is used only when it is possible to separate variables.

i.e. p & x on one side and y & q on other side.

Type IV: Clairaut's Form

Equations of the form $z = px + qy + f(p, q)$

Let the required solution be $z = ax + by + c$, then

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \quad \text{and} \quad \frac{\partial z}{\partial y} = b \Rightarrow q = b$$

\therefore Required solution is $z = ax + by + f(a, b)$

i.e. Directly substitute a in place of p and b in place of q in the given equation.

Equations Reducible to Standard Forms

Equations of the type $f(x^m p, y^n q) = 0$, where m and n are constants.

Now, let us transform above equation to the form $f(P, Q) = 0$ (Type-I)

Case-I: If $m \neq 1$ and $n \neq 1$

Put $X = x^{1-m}$ and $Y = y^{1-n}$, then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P(1-m)x^{-m} \quad (\because X = x^{1-m} \Rightarrow \frac{\partial X}{\partial x} = (1-m)x^{-m} \text{ and } P = \frac{\partial z}{\partial X})$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = Q(1-n)y^{-n} \quad (\because Y = y^{1-n} \Rightarrow \frac{\partial Y}{\partial y} = (1-n)y^{-n} \text{ and } Q = \frac{\partial z}{\partial Y})$$

Substituting these values in the given equation, we get

$$f(x^m p, y^n q) = 0 \Rightarrow f(x^m P(1-m)x^{-m}, y^n Q(1-n)y^{-n}) = 0$$

$$\Rightarrow f((1-m)P, (1-n)Q) = 0$$

which is in the form of $f(P, Q) = 0$ (Type-I)

Solve this, get the result which will be in terms of X and Y and the substitute $X = x^{1-m}$ and $Y = y^{1-n}$, which is the required solution.

Case-II: If $m = 1$ and $n = 1$

Put $X = \log x$ and $Y = \log y$, then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P \frac{1}{x} \quad (\because X = \log x \Rightarrow \frac{\partial X}{\partial x} = \frac{1}{x} \text{ and } \frac{\partial z}{\partial X})$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = Q \frac{1}{y} \quad (\because Y = \log y \Rightarrow \frac{\partial Y}{\partial y} = \frac{1}{y} \text{ and } \frac{\partial z}{\partial Y})$$

Substituting these values in the given equation, we get

$$\begin{aligned} f(x^m p, y^n q) = 0 &\Rightarrow f(xp, yq) = 0 \\ &\Rightarrow f\left(xP \frac{1}{x}, yQ \frac{1}{y}\right) = 0 \\ &\Rightarrow f(P, Q) = 0 \text{ (Type-I)} \end{aligned}$$

Solve this, get the result which will be in terms of X and Y and the substitute $X = \log x$ and $Y = \log y$, which is the required solution.

❖ Equations of the type $f(x^m p, y^n q, z) = 0$, where m and n are constants

This equation can be reduced into $f(P, Q, z) = 0$ (Type-II) by taking above substitutions.

Equations of the type $f(pz^n, qz^n) = 0$, where n is a constant

In order to convert into the form $f(P, Q) = 0$, we have to take the following substitutions

$$\text{Put } Z = \begin{cases} z^{n+1}, & \text{if } n \neq -1 \\ \log z, & \text{if } n = -1 \end{cases}$$

❖ Equations of the type $f_1(x, pz^n) = f_2(y, qz^n)$ where n is a constant.

These type of equations can be reduced to the form $f(P, Q) = 0$ (Type-I) (or)

$f_1(p, x) = f_2(y, q)$ by taking above substitutions given for the equation $f(pz^n, qz^n) = 0$

CHARPIT'S METHOD

This is a general method to find the complete integral of the non-linear PDE of the form $f(x, y, z, p, q) = 0$

Now Auxillary Equations are given by

$$\frac{dx}{\left(-\frac{\partial f}{\partial p}\right)} = \frac{dy}{\left(-\frac{\partial f}{\partial q}\right)} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{p \frac{\partial f}{\partial z} + \frac{\partial f}{\partial x}} = \frac{dq}{q \frac{\partial f}{\partial z} + \frac{\partial f}{\partial y}}$$

Here we have to take the terms whose integrals are easily calculated, so that it may be easier to solve p and q .

Finally substitute in the equation $dz = p dx + q dy$

Integrate it, we get the required solution.

Note that all the above (TYPES) problems can be solved in this method.

Method of Separation of variables

This method involves a solution which breaks up into product of functions, each of which contains only one of the independent variables.

Procedure: For the given PDE, let us consider the solution to be $z = X(x) Y(y)$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial X}{\partial x} Y = X' Y$$

$$\frac{\partial z}{\partial y} = X \frac{\partial Y}{\partial y} = X Y'$$

Substitute these values in the given equation, from which we can separate variables.

Write the equation such that X', X and x terms are on one side and similarly Y', Y and y terms are on the other side.

Let it be $F(X', X, x) = G(Y', Y, y) = \lambda$

$\Rightarrow F(X', X, x) = \lambda$ and $G(Y', Y, y) = \lambda$

Solve these equations; finally substitute in $z = X(x) Y(y)$ which gives the required solution.
